UNIVERSIDADE DE LISBOA

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Statistical solutions and invariant measures in Hydrodynamics

" Documento Definitivo"

Doutoramento em Matemática

Especialidade de Física Matemática e Mecânica dos meios contínuos

Alexandra Symeonides

Tese orientada por:

Prof. Doutor Jean-Claude Zambrini

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Documento especialmente elaborado para a obtenção do grau de doutor

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Resumo

Neste trabalho considera-se uma abordagem probabilística aos problemas da existência das soluções de modelos bidimensionais da hidrodinâmica e do estudo das suas propriedades. As soluções que procuramos são definidas quase certamente com respeito a medidas de probabilidade em dimensão infinita. Grosso modo, estas medidas fornecem a probabilidade de encontrar a dinâmica numa dada configuração e tempo. Geralmente são medidas Gaussianas definidas por operadores de covariância dados por quantidades físicas do movimento. Por esta razão, chamamos-lhes "medidas de Gibbs" e denotamo-as por

$$\text{``}d\mu(u) = e^{-S(u)}\mathcal{D}u\text{''},$$

onde $S(u)$ denota uma quantidade física do sistema e $\mathcal{D}u$ a "medida de Lebesgue em dimensão infinita" (note-se que não existe medida de Lebesgue em dimensão infinita, veja-se [49]. As medidas de Gibbs têm de ser consideradas como limite de aproximações em dimensão finita numa topologia adequada.)

À luz deste facto, soluções probabilísticas correspondem a configurações do sistema dinâmico seleccionadas com probabilidade igual a um. Consideram-se valores iniciais no suporte das medidas, geralmente conjuntos constituídos por funções muito irregulares, tipicamente distribuções (com regularidade de tipo Sobolev e de ordem negativa).

A minha motivação principal é o estudo de soluções estatísticas para equações de tipo Euler com dados iniciais no suporte de medidas invariantes ou quase-invariantes. Este estudo é baseado em [66, 34, 33] e trata de: equações averaged-Euler com condições de fronteira periódicas; equações de Euler no caso não periódico; uma versão modificada das equações de Euler. Por fim, termina-se esta tese considerando as equações de Navier-Stokes estocásticas no toro de dimensão dois, conteúdo do Capítulo 5.

O modelo de averaged-Euler descreve o movimento de um fluido não viscoso e incompressível. Consideram-se as equações em dimensão dois e no toro $\mathbb{T}^2 \simeq [0, 2\pi]^2$, estas são dadas por

$$\frac{\partial Au}{\partial t} + (u \cdot \nabla)Au + (\nabla u)^T \cdot Au = -\nabla p, \qquad \nabla \cdot u = 0, \tag{1}$$

onde $A = (1-a^2\Delta)^s$ com a um parâmetro real e s um número positivo. A velocidade média do fluido é denotada por $u : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ e a pressão por $p : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$. Estas equações foram introduzidas pelos autores de [44] com a intenção de considerar uma modificação do modelo de Euler, de forma a que efeitos não lineares em escalas pequenas do movimento sejam negligíveis. Isto implica que a dinâmica continua turbulenta, mas não dissipativa.

Por estas equações construí medidas do tipo Gibbs com respeito à enstrofia,

$$S(\varphi) := \frac{1}{2}\int_{\mathbb{T}^2} (A\Delta\varphi)^2 dx,$$

onde φ denota a "stream function", veja-se Section 2.2. Dado $\gamma \in \mathbb{R}^+$, definimos estas medidas como

$$d\mu_\gamma(\varphi) = \prod_{k>0} \frac{\gamma k^4(1+a^2k^2)^{2s}}{2\pi} \exp\left\{-\frac{1}{2}\gamma k^4(1+a^2k^2)^{2s}|\varphi_k|^2\right\} d\varphi_k,$$

onde φ_k denota a k-ésima componente da expansão de Fourier de φ na base ortonormal de $L^2([0,2\pi]^2)$, dada pelas funções próprias do operador de Laplace com condições de fronteira periódicas. Sendo a enstrofia uma quantidade conservada pelo movimento, estas medidas são formalmente invariantes pelo fluxo da equação averaged-Euler. Efectivamente, demonstramos que a divergência do campo de vectores da equação averaged-Euler com respeito a esta medida é nula e também provamos que o campo é $L^p_{\mu_\gamma}$-integrável. Estes factos permitem provar a existência de soluções estatísticas para as quais as medidas μ_γ são invariantes.

Outras quantidades conservadas podem ser usadas com o mesmo propósito de definir medidas invariantes, no caso da energia E denotamos estas medidas por $\mu_{\gamma,E}$. Por um lado, o campo de vectores averaged-Euler não é de quadrado integrável com respeito às $\mu_{\gamma,E}$, portanto não é possível construir soluções probabilísticas com respeito a estas. Por outro lado, a energia é suficientemente regular para definir uma outra medida (desta vez não necessariamente Gaussiana) sobre os conjuntos de nível de E e para construir sobre estes uma velocidade averaged-Euler.

Por fim, demonstramos que a solução é recorrente, i.e. saindo de qualquer ponto inicial no suporte, volta quase certamente e infinitas vezes numa vizinhança do valor inicial.

As *equações de Euler no plano* são dadas por:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \qquad \nabla \cdot u = 0 \tag{2}$$

onde $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ denota o campo de velocidade média e $p : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ a pressão. A primeira equação corresponde à segunda lei de Newton (a aceleração é directamente proporcional à pressão) e a segunda é a condição de incompressibilidade.

Em [3] foi estudada a existência de um fluxo probabilístico para as equações de Euler em dimensão dois e sobre o toro $[0,2\pi]^2$. Reescalando, podemos considerar estes fluxos no espaço de fase $[0,L]^2$ e tratar da existência no plano, considerando o limite do período L que tende para o infinito. De facto, podemos definir os processos estocásticos $\{\Phi_L\}_{L\in\mathbb{N}^*}$ (com valores no espaço de Sobolev H^β por $\beta < 1$) cujas leis são as medidas invariantes construídas em [3] (depois de reescalar) e que denotamos por $\mu_{L,\gamma}$. Demonstramos que $\{\Phi_L\}_{L\in\mathbb{N}^*}$ é uma sucessão de Cauchy em $L^2(\Omega; H^\beta_{loc}(\mathbb{R}^2))$ o que implica que as $\mu_{L,\gamma}$ convergem fracamente para uma certa medida μ_γ com respeito à topologia de $H^\beta_{loc}(\mathbb{R}^2)$. O espaço $H^\beta_{loc}(\mathbb{R}^2)$ é o suporte de μ_γ por valores de β menores que um. Finalmente, provamos a existência de um fluxo integral e mostramos que este é continuo de $H^\beta_{loc}(\mathbb{R}^2)$ para $H^\beta_{loc}(\mathbb{R}^2)$ no suporte de μ_γ por todos $t \in \mathbb{R}$.

Neste trabalho também consideramos as *equações de Euler no plano com uma modi-*

ficação na contribuição dada pela pressão

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla)\tilde{u} = -\nabla p + cxp, \qquad \mathrm{div}\tilde{u} = 0 \tag{3}$$

onde a pressão $p : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ pode depender de c e c é um parâmetro fixado em $(0, 1)$. Depois da mudança de variáveis

$$u(t, x) = \sigma^c(x)\tilde{u}(t, x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^2, \tag{4}$$

onde $\sigma^c(x) = \frac{1}{2\pi}e^{-\frac{c|x|^2}{2}}$ denota a densidade Gaussiana em $\mathbb{R}^2$, a equação lê-se

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)(\rho^c u) = -\nabla(\sigma^c p), \qquad \mathrm{div}_{\rho^c} u = 0, \tag{5}$$

sendo que $\rho^c(x) := (\sigma^c)^{-1}(x) = 2\pi e^{\frac{c|x|^2}{2}}$ e $\mathrm{div}_{\rho^c} u$ é definida por

$$\int_{\mathbb{R}^2} \mathrm{div}_{\rho^c} u f d\rho^c = -\int_{\mathbb{R}^2} u \cdot \nabla f d\rho^c, \qquad \forall f \in \mathcal{C}_c^1$$

(usamos a notação $d\rho^c = \rho^c dx$). Assumimos que as condições iniciais de (5) são dadas por $u_0 = \sigma^c\tilde{u}_0$, onde $\tilde{u}_0$ é a condição inicial de (3), e que $\tilde{u}$ e u convergem para zero no infinito. Veremos que esta mudança de variáveis permite-nos estudar as equações em $L^2_{\sigma^c}(\mathbb{R}^2)$, o espaço das funções com valores reais e de quadrado integrável em relação à medida $\sigma^c dx$. Tal como as equações de Euler, esta modificação possui infinitas quantidades conservadas. Este facto permite-nos provar a existência de soluções fracas (no sentido clássico) e no mesmo espírito do trabalho de Judovich para Euler [46]. A vantagem de usar esta modificação advém de termos uma base ortonormal de funções próprias do operador de Ornstein-Uhlenbeck (os polinómios de Hermite). Efectivamente, a formulação da equação em termos da vorticidade corresponde agora à clássica, mas com o operador de Ornstein-Uhlenbeck em lugar do de Laplace,

$$\frac{\partial}{\partial t} L^c\varphi = -(\nabla^{\perp}\varphi \cdot \nabla)L^c\varphi. \tag{6}$$

Definimos medidas $\mu_{\sigma^c,\gamma}$ cujos suportes contêm funções regulares e não só distribuições, estas são $L^p_{loc}(\mathbb{R}^2)$ para cada $p \in (2, 10/3)$. Depois de estudar a $L^r_{\mu_{\sigma^c,\gamma}}$-regularidade do campo vectorial, das suas derivadas e da divergência, provamos a existência de um único fluxo pelo qual as $\mu_{\sigma^c,\gamma}$ são quase-invariantes. É claro que a solução construída é uma boa aproximação (por "pequenas" modificações) da velocidade Euler no plano. Contudo, ao variar do parâmetro c (quando c converge para zero) não podemos considerar o limite das soluções construídas, sendo que estas são definidas quase certamente com respeito às medidas $\mu_{\sigma^c,\gamma}$ (que também dependem do parâmetro), que são singulares uma em relação a outra.

Por fim, as *equações de Navier-Stokes estocásticas* em $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$ descrevem o movimento de um fluido incompressível e viscoso

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + \varepsilon \Delta u - \nabla p + \dot{B}_t, \qquad \nabla \cdot u = 0,$$

onde $\varepsilon > 0$ denota o coeficiente de viscosidade e $\dot{B}_t$ a perturbação dada pela derivada formal de um movimento Browniano cilíndrico e renormalizado.

Destas equações estudamos o limite quando a viscosidade ε converge para zero. Provamos a existência de uma subsucessão fracamente convergente. A demonstração deste resultado baseia-se, em particular, na invariância de uma certa medida de probabilidade μ pelo fluxo e o facto deste ser uniformemente limitado.

Palavras chaves: sistemas dinâmicos aleatórios em dimensão infinita; medidas invariantes; medidas quase-invariantes; equações de Euler; equações de Navier-Stokes estocásticas.

Abstract

This thesis concerns a probabilistic approach to the problem of existence of solutions for two-dimensional models in hydrodynamics and the study of their properties. The solutions we refer to are almost everywhere defined with respect to infinite-dimensional probability measures. Roughly speaking, these measures give the probability of finding the dynamics in a certain configuration at a given time. In light of this, probabilistic solutions correspond to configurations of the dynamical system selected with probability one. Initial data belong to the support of the measures (consisting typically of irregular functions).

We recall previous results about existence and uniqueness of infinite-dimensional random dynamical systems and present the two-dimensional models from hydrodynamics considered in this thesis: periodic averaged-Euler equations; non-periodic Euler equations; a modification of the Euler equation and stochastic Navier-Stokes equations.

For the two-dimensional averaged-Euler equation we define a Gaussian invariant measure and show the existence of its solution with initial conditions on the support of the measure. An invariant surface measure on the level sets of the energy is also constructed, as well as the corresponding flow. Poincaré recurrence theorem is used to show that the flow returns infinitely many times in a neighborhood of the initial state.

For the 2D Euler equation on the plane we construct Gaussian invariant measures. We obtain them as the weak limit of those previously considered in [3] for the torus. We show the existence of solution with initial conditions on the support of the measures. Continuity of the velocity flow is proved.

Also, we consider a modified Euler equation on $\mathbb{R}^2$. We prove existence of weak global solutions for bounded (and fast decreasing at infinity) initial conditions and construct Gibbs-type measures on function spaces which are quasi-invariant for the modified Euler flow. Almost everywhere with respect to such measures (and, in particular, for less regular initial conditions), the flow is shown to be globally defined.

Finally we study the limit of a perturbed Navier-Stokes flow when the viscosity coefficients converges to zero. We show the existence of a weak limit.

Keyword: infinite-dimensional random dynamical systems; invariant measures; quasi-invariant measures; Euler equations; stochastic Navier-Stokes equations.

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Contents

Chapter 1

Introduction

1.1 Invariant and quasi-invariant measures

This thesis is about a probabilistic approach to the problem of existence of solutions for two-dimensional models in hydrodynamics and the study of their properties. These include the Euler and the averaged-Euler equations, for which both the periodic and non-periodic cases are considered. The solutions we refer to are said statistical or probabilistic, since they are almost everywhere defined with respect to (infinite-dimensional) probability measures. Roughly speaking, these measures give the probability of finding the dynamics in a certain configuration at a given time, usually they are Gaussian distributed with covariance operators given by physical quantities of the motion. Therefore we refer to these measures as Gibbs measures and, formally, we denote them by

$$\text{“}d\mu(u) = e^{-S(u)}\mathcal{D}u\text{”},$$

where $S(u)$ denotes a physical quantity of the motion and $\mathcal{D}u$ is the infinite-dimensional flat measure[1]. In light of this, probabilistic solutions correspond to configurations of the dynamical system selected with probability one. Initial data belong to the support of the measures and typically these sets are made of irregular functions (with Sobolev regularity), usually distributions (with Sobolev regularity of negative order).

Fortunately, Gaussian measures make sense in infinite dimensional spaces. Indeed they are invariant by the rotations of another Hilbert space which is embedded in the original one.

Definition 1.1.1. Let H denote a separable Hilbert space with norm $|\cdot| = \sqrt{<\cdot,\cdot>}$ and $\mathcal{F}$ be the partially ordered set of finite-dimensional orthogonal projections P of H ($P < Q$ means $P(H) \subset Q(H)$ for P, Q in $\mathcal{F}$).

[1] There exists no infinite-dimensional Lebesgue measure, see [49]. The measures must be considered as limit of finite-dimensional approximations in a suitable topology.

i) A subset E of H is called a *cylinder set* if has form $E = \{x \in H : Px \in F\}$, where $P \in \mathcal{F}$ and F is a Borel subset of $P(H)$. With $\mathcal{R}$ we denote the collection of cylinder sets.

ii) A *Gaussian measure* μ in H is the set function μ from $\mathcal{R}$ into $[0, +\infty)$ defined by

$$\mu(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_F e^{-\frac{|x|^2}{2}} dx, \qquad \forall\ E \in \mathcal{R}$$

where $n = \dim P(H)$ and dx is the Lebesgue measure of $P(H)$.

iii) A seminorm $\|\cdot\|$ in H is called *measurable in the sense of Gross* if for every $\varepsilon > 0$ there exists a $P_0 \in \mathcal{F}$ such that $\mu\{\|Px\| > \varepsilon\} < \varepsilon$ for all $P \perp P_0$ and $P \in \mathcal{F}$.

iv) The triple (X, H, μ) denotes an *abstract Wiener space*, if X is the closure of H with respect to the measurable norm in the sense of Gross $\|\cdot\|_X$ and μ is a Gaussian measure. Here X denotes the support of the measure μ and H the Cameron-Martin space which is, by construction, the space under which μ is translation (quasi) invariant and such that $\mu(H) = 0$.

Example 1. *The classical Wiener space $C([0, 1])$ is an example of abstract Wiener space and can be constructed in this way, see [49].*

Now we recall the definitions of invariant and quasi-invariant measures. For a certain dynamics

$$\begin{cases} \frac{d}{dt} U_t(x) = \Psi(U_t(x)) \\ U_0(x) = x \end{cases}$$

and as long as the flow U_t exists, a measure can be invariant or quasi-invariant under it. This is, respectively, the case in which the push-forward of the measure under the flow is the measure itself or is only absolutely continuous with respect to it.

Definition 1.1.2. Consider a measure μ, let U_t be a μ-measurable flow for all $t \in \mathbb{R}$ and denote by $\mathcal{D}$ a suitable test functions space. [2] Then

i) μ is said to be *invariant* under the flow, if $dU_t * \mu = d\mu$ for all $t \in \mathbb{R}$. That is

$$\int_X f(U_t(x)) d\mu(x) = \int_X f(x) d\mu(x), \quad \forall\ f \in \mathcal{D} \text{ and } \forall\ t \in \mathbb{R};$$

ii) μ is said to be *quasi-invariant* under the flow, if $dU_t * \mu = k_t d\mu$ for all $t \in \mathbb{R}$. That is

$$\int_X f(U_t(x)) d\mu(x) = \int_X f(x) k_t(x) d\mu(x), \quad \forall\ f \in \mathcal{D} \text{ and } \forall\ t \in \mathbb{R}.$$

[2]Below $\mathcal{D}$ will denote the space of sufficiently regular functions depending on a finite number of coordinates, namely it will denote a space of cylindrical functions.

Thanks to the invariance property, a local in time existence result may be extended to a global one. The same holds for quasi-invariant measures, if the corresponding densities are uniformly bounded in suitable normed spaces. Below we give other useful definitions from Malliavin calculus.

Definition 1.1.3. On an abstract Wiener space (X, H, μ), given a field $\Psi : X \to G$, where X is a Banach space and G a separable Hilbert space,

i) the *gradient* (in the sense of Malliavin calculus [53]) of Ψ is defined for every $u \in X$ by
$$\nabla\Psi(u)(v) = D_v\Psi(u) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\Psi(u+\varepsilon v) - \Psi(u)], \quad v \in H$$
and the limit is taken μ-a.e. in X.

ii) The *divergence* of $\Psi \in L^2_\mu(X; G)$ is denoted by $\delta_\mu\Psi$ and defined by
$$\int_X \delta_\mu\Psi \cdot f d\mu = -\int_X (\Psi, \nabla f)_G d\mu, \quad \forall\, f \in \mathcal{D} \tag{1.1}$$
where $\mathcal{D}$ is the space of cylindrical functions on X and $(\cdot,\cdot)_G$ denotes the inner product of G. The divergence of Ψ with respect to μ is therefore the adjoint of the gradient operator in L^2_μ.

iii) We say that μ is *infinitesimally invariant* for Ψ if
$$\int_X D_h\Psi \; d\mu = 0, \quad \forall\, h \in H.$$
In particular, if the divergence of Ψ with respect to μ is equal to zero, then μ is infinitesimally invariant for Ψ.

If there exists a flow U_t solving the dynamical system in $\mathbb{R} \times X$
$$\begin{cases} \frac{d}{dt}U_t(x) = \Psi(U_t(x)) \\ U_0(x) = x, \end{cases} \tag{1.2}$$
then $\delta_\mu\Psi = 0$ also means that $\frac{d}{dt}\Big|_{t=0} \int_X f(U_t(x))d\mu(x) = 0$ for every test function f, that is μ is invariant under U_t according to Definition 1.1.2 i). Actually, the weaker property $\delta_\mu\Psi = 0$ combined with the regularity results for Ψ allows to prove existence of an integral flow for (1.2), we will further explain this fact in the next subsection.

Intuitively and at least for Hamiltonian systems [3], infinitesimally invariance is a consequence of Liouville theorem and the fact that the covariance operator is a conserved

[3] Since the models we will work with are not written in their Hamiltonian form, below we will check that $\mathrm{div}\Psi = 0$.

quantity of the motion. Indeed the following holds in distributional sense

$$\begin{aligned}\delta_\mu \Psi &= \mathrm{div}\Psi + < \frac{\nabla \rho}{\rho}, \Psi >_G \\ &= \mathrm{div}\Psi - \frac{d}{dt} S(u) = 0,\end{aligned}$$

where ρ denotes the Radon-Nikodym density of μ with respect to the “Lebesgue measure” and divΨ the divergence with respect to the “Lebesgue measure”. Next, we will see that under some exponential integrability assumption on the divergence of Ψ the measure is quasi-invariant [30]. Sometimes the supports of quasi-invariant measures consist of more regular functions and regularity issues may be improved [68, 57]. In this thesis, namely in [33], we construct quasi-invariant measures with regular supports as a consequence of working in Gaussian weighted Sobolev space.

1.1.1 Existence of the flow

The study of existence of almost everywhere defined flows for systems with very irregular vector fields (with Sobolev or BV regularity), started in [30] and was successively extended by [70, 10, 40]. In [30] existence follows from a compactness argument valid under exponential integrability assumptions for the vector field, its derivatives and its divergence.

Theorem 1.1.1 (A. B. Cruzeiro 1983)**.** *Let $\Psi : X \to H$ be a vector field such that:*

1. $\Psi \in \cap_r W^{r,p}$ [4] *and for all* $\lambda > 0$ $\int_X e^{\lambda \|\Psi(x)\|_H} d\mu(x) < \infty$*;*
2. *for all* $\lambda > 0$, $\int_X e^{\lambda \|\nabla \Psi(x)\|_{H.S.(H;H)}} d\mu(x) < \infty$[5]*;*
3. *for all* $\lambda > 0$, $\int_X e^{\lambda |\delta_\mu \Psi(x)|} d\mu(x) < \infty$*;*

Then,

i) there exists U_t verifying the equation

$$\begin{cases} \frac{d}{dt} U_t(x) = \Psi(U_t(x)) \\ U_0(x) = x, \end{cases}$$

for all times $t \in \mathbb{R}$ and for μ- a.e. $x \in X$.

ii) The measure $U_t * \mu$ is absolutely continuous with respect to μ and, if we denote $k_t(x) = \frac{dU_t * \mu}{d\mu}(x)$, we have $k_t(x) \in L^p_\mu$ for all p.

iii) If $\delta_\mu \Psi \in W^{1,16}$, we also have $k_t(x) = e^{\int_0^t \delta_\mu \Psi(U_s(x)) ds}$.

[4]We denote by $W^{r,p}$ the space of functions with Malliavin derivatives up to r-th order in L^p_μ.
[5]We denote by $H.S.(H;H)$ the space of Hilbert-Schmidt operators from H to H.

This result was improved by A. S. Üstunel [70] and was employed in [33] which is part of this thesis.

Theorem 1.1.2 (A. S. Üstunel 2000)**.** *Let $\Psi : \mathbb{R}^+ \times X \to H$ be a measurable map such that $t \mapsto \mathbb{E}_\mu \|\Psi(t,x)\|_H$ is locally integrable; $\delta_\mu \Psi(t,\cdot) \in L^p_\mu(\mathbb{R})$ for some $p > 1$ and $< \Psi(t,\cdot), h >_H \in W^{1,p}$ for any $h \in H$ and almost all $t \in \mathbb{R}^+$. Assume that, for given $T > 0$, there exists some $\varepsilon_0 > 0$ such that*

$$\int_0^T \mathbb{E}_\mu \left[\exp\{\varepsilon_0 |\delta_\mu \Psi(s,x)|\} + \exp\{\varepsilon_0 \|\nabla \Psi(s,x)\|\}\right] ds < +\infty$$

where $\|\nabla \Psi(s,x)\| = \sup_{|h|_H \leq 1} |D_h \Psi(s,x)|_H$. Then there exists a family of measurable transformations $\{U_{st}(x)\}_{0 \leq s \leq t \leq T}$ of X such that $dU_{st} * \mu$ is equivalent to μ for every $s < t$ in $[0,T]$ such that

$$U_{st}(x) = x + \int_s^t \Psi(r, U_{sr}(x))dr, \quad \mu - a.e.\ x \in X \text{ and } \forall\ t \in [0,T].$$

Moreover the family $\{U_{st}(x)\}_{0 \leq s \leq t \leq T}$ is a flow of invertible transformations. The paths $t \mapsto U_{st}$ are μ-a.s. continuous on $[s,T]$, as X-valued trajectories, for any $s \in [0,T]$. The Radon-Nikodym densities are given by

$$\frac{dU_{st} * \mu}{d\mu}(x) = \exp\left\{\int_s^t |\delta_\mu \Psi(r, U_{rt}^{-1}(x))|dr\right\}.$$

Besides, for any $p > 1$ with $t - s < \frac{\varepsilon_0}{p}$, we have

$$\mathbb{E}_\mu \frac{dU_{st} * \mu}{d\mu}(x) \leq \frac{e^{\frac{1}{p^2}}}{q\varepsilon_0} \mathbb{E}_\mu \int_s^t \exp\{\varepsilon_0 |\delta_\mu \Psi(r,(x))|\}\, dr$$

for q the conjugate exponent of p. Finally U_{st} with the above properties is unique.

In [10, 40] a kind of DiPerna-Lions theory [6] for flows associated to Sobolev vector fields is extended to the case of Cameron-Martin valued vector fields in Wiener spaces having Sobolev regularity.

Definition 1.1.4. Let $\Psi : (0,T) \times X \to X$ be a Borel vector field. If $U : [0,T] \times X \to X$ is Borel and $1 \leq r \leq \infty$, we say that U is a L^r*-regular flow* associated to Ψ if the following two conditions hold:

i) for μ-a.e. $x \in X$ the map $t \mapsto \|\Psi(t, U(t,x))\|_X$ belongs to $L^1(0,T)$ and

$$U(t,x) = x + \int_0^t \Psi(s, U(s,x))ds, \quad \forall\ t \in [0,T]; \tag{1.3}$$

[6] Well-posedness for the ODE is proven once well-posedness for the associated continuity equation (instead of the transport equation) is shown.

ii) for all $t \in [0,T]$ the law of $U(t,\cdot)$ under μ is absolutely continuos with respect to μ, with a density $k_t \in L^r_\mu$ and $\sup_{t\in[0,T]} \|k_t\|_{L^r_\mu} < \infty$.

Here we recall the well-posedness result from [10].

Theorem 1.1.3 (L. Ambrosio, A. Figalli 2009)**.** *Let $p, q > 1$ and $\Psi : (0,T) \times X \to H$ be such that:*

i) $\|\Psi(t,\cdot)\|_H \in L^1((0,T); L^p_\mu)$;

ii) for a.e. $t \in (0,T)$ we have $\Psi(t,\cdot) \in LD^q_H(\mu; H)$ with

$$\int_0^T \left(\int_X \|(\nabla\Psi(t,x))^{sym}(x)\|_{H.S.(H;H)} d\mu(x) \right)^{1/q} dt < \infty, \tag{1.4}$$

and $\delta_\mu \Psi(t,\cdot) \in L^1((0,T); L^q_\mu)$;

iii) $\exp \varepsilon_0 [\delta_\mu \Psi(t,\cdot)]^- \in L^\infty((0,T); L^1_\mu)$ *for some $\varepsilon_0 > 0$.*

If $r := \max\{p', q'\}$ and $\varepsilon_0 \leq rT$, then the L^r-regular flow exists and is unique in the following sense: any two L^r-regular flows U and $\tilde{U}$ satisfy

$$U(\cdot, x) = \tilde{U}(\cdot, x), \quad \text{in } [0,T], \text{ for } \mu - a.e.\ x \in X.$$

Furthermore, U is L^r-regular for all $s \in [1, \frac{\varepsilon_0}{T}]$ and the density k_t of the law of $U(t,\cdot)$ under μ satisfies

$$\int_X (k_t(x))^s d\mu(x) \leq \left\| \int_X \exp\left(Ts[\delta_\mu \Psi(t,x)]^-\right) d\mu(x) \right\|_{L^\infty(0,T)}, \quad \forall\ s \in \left[1, \frac{\varepsilon_0}{T}\right].$$

In particular, if $\exp \varepsilon_0 [\delta_\mu \Psi]^- \in L^\infty((0,T); L^1_\mu)$ for all $\varepsilon_0 > 0$, then the L^r-regular flow exists globally in time, and is L^s-regular for all $s \in [1, \infty)$.

The matrix $\nabla\Psi^{\mathrm{sym}}$ denotes the symmetric part of the weak derivatives of Ψ which jointly with the spaces $LD^q_H(\mu; H)$ are defined in Definition 2.6 of [10]. For our purpose, we only need to remark that the spaces $LD^q_H(\mu; H)$ contain $W^{1,q}$ which in turn are the spaces we will work with. In particular, if all the components of Ψ belong to $W^{1,q}$ then the functions $(\nabla\Psi^{\mathrm{sym}})_{i,j}$ are, in fact, the components of the symmetric part of $(\nabla\Psi)_{i,j}$.

Remark 1.1.1. Since the spaces are infinite-dimensional, in [10, 40, 70, 30] the following limitation is taken into account: the vector field Ψ must take values in the Cameron-Martin space H in order for the measure μ to be quasi-invariant. For example, the field defined by $\Psi(t,x) = x + tv$ for a given v only leaves μ quasi-invariant when v belongs to H. For the particular cases of the Euler and averaged-Euler vector fields [66, 34], we do not have enough regularity to apply the existence results established in [10, 70, 30, 40].

Last, different resolution methods have been employed for each particular model: regarding nonlinear Hamiltonian PDEs, we refer to the works of J. Bourgain about the nonlinear Schrödinger equations (NLS) [17, 18]; of N. Tzvetkov and collaborators about the Benjamin-Ono [69] and NLS [57] equations and of A.-S. de Suzzoni about the Klein-Gordon equations [35] or NLS equations [25].

In the context of hydrodynamics, namely concerning Euler and Navier-Stokes equations we recall the works [4, 3, 27]. Where, respectively, invariant measures for the 2D Euler equations are constructed; invariant measures are used to construct probabilistic flows for the 2D Euler equations and for the stochastically perturbed 2D Navier-Stokes equations; conditional invariant measures [7] are constructed such that flows exist on the level sets of a suitable "renormalized" energy. [8]

The strategy for the proof of the existence that we will use in this work is the one used by S. Albeverio and A. B. Cruzeiro in [3]. It relies on the application of Prokhorov and Skorokhod's theorems to a sequence of Gibbs-type probability measures which are infinitesimally invariant under some approximating flows, in order to get tightness of the measures and consequently existence of an integral flow. Additionally, this theory was followed in [25] and revisited by F. Flandoli in [41] and subsequent works, following the approach of weak vorticity formulation. In [66, 34] (which are part of this thesis) this method was used, thus we recall it here jointly with the statements of Prokhorov and Skorokhod's theorems. Proofs can be found, respectively, in [65] and [45].

Theorem 1.1.4 (Y. V. Prokhorov 1956)**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on a complete separable metric space X and assume that for each $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset X$ such that*

$$\inf_{n\in\mathbb{N}} \mu_n(K_\varepsilon) \geq 1 - \varepsilon.$$

Then $\{\mu_n\}_{n\in\mathbb{N}}$ is pre-compact in the space of Borel probability measures on X equipped with the weak-topology.

Theorem 1.1.5 (A. V. Skorokhod 1956)**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on a complete separable metric space X converging weakly to a Borel measure μ. Then there exist a probability space $(\Omega, \mathbb{P})$ and measurable mappings $\xi_n, \xi : \Omega \to X$ such that $\mu_n = \mathbb{P} \circ \xi_n^{-1}$, $\mu = \mathbb{P} \circ \xi^{-1}$ and $\xi_n \to \xi$ $\mathbb{P}$-almost everywhere.*

[7] Infinite-dimensional surface or conditional measures were formally defined in [1], see Appendix A.

[8] The energy is not integrable and it is not possible to use results on existence of solutions of the Euler equation with initial conditions of finite energy to construct a flow. Therefore a "renormalized" energy is defined.

Existence of an Euler velocity on $\mathbb{T}^2$: Strategy of the proof [3].

1. Solve the finite dimensional approximated equation; a globally defined solution exists by classical results since the Euler fields Ψ^n are quadratic and the energy is conserved by the motion (see [3]). Hence there exists an integral flow $U_t^n(x) = x + \int_0^t \Psi^n(s, U_s^n(x))ds$ defined almost everywhere and for all times.

2. Consider the flow maps $\{U_t^n\}_{n\in\mathbb{N}}$ as stochastic processes with laws ν_n in $C(\mathbb{R};\mathbb{R})$ and show, by Prokhorov theorem, that these laws are pre-compact, that is there exists a subsequence $\nu_{n,j}$ converging weakly to some ν.

3. By Skorokhod theorem, conclude that there exists a probability space $(\Omega, \mathbb{P})$ and stochastic processes $\tilde{U}_t^n, \tilde{U}_t$ with laws respectively ν_n, ν on $(\Omega, \mathbb{P})$ and such that $\tilde{U}_t^n \to \tilde{U}_t$ $\mathbb{P}$-a.e. $\omega \in \Omega$ and for all times.

4. Last, show that $\tilde{U}_t$ takes values almost surely in X and that is an integral flow for the Euler field in the following sense:

$$\tilde{U}_t(\omega) = x + \int_0^t \Psi(\tilde{U}_s(\omega))ds, \quad \mathbb{P} - a.e.\ \omega \in \Omega, \forall\ t \in \mathbb{R}. \tag{1.5}$$

   This fact is easy to prove once it is know that $\delta_\mu \Psi = 0$ (and consequently that the probability measure μ is invariant under $\tilde{U}_t$) and that $\{\Psi^n\}_{n\in\mathbb{N}}$ are equi-integrable.

Remark 1.1.2. The statistical solutions obtained by the scheme presented here are in fact stochastic processes defined for μ-a.e. initial data in the support of the measures.

Remark 1.1.3. By the same method, in [3], a periodic and two-dimensional stochastic Navier-Stokes flow is constructed. We will be back on this in Chapter 5, where asymptotics for a perturbation of the Navier-Stokes equation is studied.

In this work we will use the approach by S. Albeverio e A. B. Cruzeiro in the case of the averaged-Euler equations [66] and for the non-periodic Euler equations [34], while A. S. Üstunel's result was used for the more regular modified Euler equations [33].

1.1.2 Uniqueness

Since the earliest works about the constructions of invariant measures and of the corresponding probabilistic solutions for the Euler equation [4, 8, 3], uniqueness remains an open problem (see [7]). In [8] S. Albeverio and R. Høegh-Khron consider the Liouville operator L associated to the Euler vector field B, that is the symmetric linear operator in L^2_μ defined by $L = B/i$ and with domain given by the subspace of cylinder smooth bounded functions. There exist self-adjoint extensions of L and they observed that if this extension is unique (i.e. if L is essentially-self adjoint), then uniqueness for the Euler equation holds. In [5] S.

Albeverio and B. Ferrario showed that this operator is bounded by a naturally associated (positive Schrödinger-like) operator which is essentially self-adjoint on a dense subspace of cylinder functions. A uniqueness result on extensions of L in a space different from L^2_μ has been obtained in [2]. Contrarily, pathwise uniqueness[9] of solutions of the stochastic Navier-Stokes equation is easier and was proved in [6].

Below we discuss some results about pathwise uniqueness from another point of view, namely by the infinite-dimensional DiPerna-Lions theory for flows associated to weakly differentiable vector fields (see [10]). In this framework well-posedness for the differential equation holds if and only if it holds for the corresponding continuity equation, see Proposition 1.1.1 below. Uniqueness of solutions of the continuity equation is a delicate matter and counterexamples exist even for the simplest divergence-free case (see Depauw's counterexample presented in G. Crippa PhD thesis [29]).

We stress that the main difference between our situation with respect to the one in [10, 70, 31] is that we are not dealing with Cameron-Martin valued vector fields. For drifts valued in any separable Hilbert space, the stability estimate given in [9] (Theorem 1.1.7 below) may be used to get uniqueness. This will not be the case for the Euler (or averaged-Euler) field, indeed the derivative used there is not the usual Malliavin one (Definition 1.1.3-i) above).

Definition 1.1.5. Let $1 \leq p < \infty$ and $\Psi \in L^p_\mu$. We say that $\Psi \in W^{1,p}(X,\mu;X)$ if for every $v \in X$ the function $< \Psi, v >_X$ belongs to $W^{1,p}(X;\mu)$[10] and

$$|\nabla\Psi|(x) := \sqrt{\sum_k |\nabla\Psi_k(x)|^2} \in L^p_\mu$$

where we denoted by Ψ_k the components of Ψ with respect to an Hilbert basis of X. We observe that $|\nabla\Psi|$ does not depend on the choice of the basis (in fact, it is the Hilbert-Schmidt norm on $X \otimes X$).

In particular, in Definition 1.1.5 are taken into account derivatives along all directions of X and therefore $W^{1,p}(X,\mu;X) \subset W^{1,p}$. We refer to [59] for further results.

Definition 1.1.6. Let X be a separable Hilbert space and fix $T > 0$, then:

1. we denote by $\Omega(X)$ the space of continuous maps from $[0,T]$ to X endowed with the supremum norm. Since X is separable, then $\Omega(X)$ is separable and complete.

2. By $e_t : \Omega(X) \to X$ we denote the evaluation map at time $t \in [0,T]$, that is $e_t(w) := w(t)$ for $w \in \Omega(X)$.

[9]By pathwise uniqueness we mean that two statistical solutions coincide if they are indistinguishable in the sense of stochastic calculus.

[10]$W^{1,p}(X;\mu)$ denotes the Sobolev space obtained as the closure of smooth cylindrical functions with respect to the norm $\|u\|_{W^{1,p}_\mu} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$ where ∇ is the usual gradient.

3. By $AC^\alpha(X) \subset \Omega(X)$ for $1 \le \alpha \le \infty$ we denote the subspace of functions w such that

$$w(t) = w(0) + \int_0^t g(s)ds, \quad \forall\, t \in [0,T]$$

for some $g \in L^\alpha([0,T];X)$. This function g is only determined up to negligible sets, indeed $< e^*, g(\bar{t}) >$ coincides with the derivative at $t = \bar{t}$ of the real-valued absolutely continuous function $t \mapsto < e*, w(t) >$ for all $e^* \in X^*$.

Definition 1.1.7. We say that a positive finite measure $\eta \in \Omega(X)$ is a *generalised* Ψ*-flow* if

1. η is concentrated on paths $w \in AC^1(X)$ such that $\dot{w} = \Psi(w)$ and

$$w(t) = w(0) + \int_0^t \Psi_s(w(s))ds, \quad \forall\, t \in [0,T];$$

2. $e_0 * \eta = \mu$.

3. For all $t \in [0,T]$, the image measures $e_t * \eta$ are absolutely continuous with respect to μ with a density in $k_t \in L^1_\mu$ and such that $e_t * \eta \le L\mu$ for some finite constant L called *compressibility constant.*

Continuity equation. We shall assume that $\mu_t := k_t\mu$ for some density function k_t to determine and we consider

$$\frac{d}{dt}\mu_t + \delta_\mu(\Psi_t\mu_t) = f\mu_t, \quad \text{in } (0,T)\times X \tag{1.6}$$

in the weak sense; namely we require that $t \mapsto \int_X g d\mu_t$ is absolutely continuous in $(0,T)$ and

$$\frac{d}{dt}\int_X g\mu_t = \int_X < \Psi_t, \nabla g >_X d\mu_t + \int_X gfd\mu_t, \qquad \text{a.e. in } (0,T),\ \forall\, g \in Cyl(X;\mu). \tag{1.7}$$

The minimal requirements to give meaning to (1.7) is that k_t, f and $|k_t|\|\Psi_t\|_X$ are $L^1((0,T);L^1_\mu)$. We will consider the source term f to be null. The assumption that $t \mapsto \int_X gk_td\mu$ is absolutely continuous in $(0,T)$ implies that $t \mapsto k_t$ is weakly continuous in $(0,T)$ with respect to the duality L^1_μ with $Cyl(X;\mu)$. Therefore it makes sense to say that a solution k_t of the continuity equation starts from $\bar{k} \in L^1_\mu$ at time zero:

$$\lim_{t\downarrow 0}\int_X gk_td\mu = \int_X g\bar{k}d\mu, \quad \forall\, g \in Cyl(X;\mu). \tag{1.8}$$

The relation between the ODE $\dot{x}(t) = \Psi_t(x(t))$ and the continuity equation $\frac{d}{dt}\mu_t + \delta_\mu(\Psi_t\mu_t) = 0$ is classical and is the issue of the following proposition.

Proposition 1.1.1. *Let η be a positive finite measure in $\Omega(X)$ satisfying:*

1. *η is concentrated on paths $w \in AC^1(X)$ such that $\dot{w} = \Psi(w)$ and*
$$w(t) = w(0) + \int_0^t \Psi_s(w(s))ds, \quad \forall\, t \in [0,T];$$

2. *$\int_0^T \int_{\Omega(X)} \|\dot{w}(t)\|_X d\eta(w)dt < \infty$.*
Then the measure $\mu_t := e_t * \eta$ satisfy the continuity equation (1.6) (with null source term) in the weak sense on $(0,T) \times X$.

In [10] is proved that uniqueness for the continuity equation (in the class of generalised Ψ-flows) implies uniqueness (in the sense specified below) of the flow and that the latter is distributed as a Dirac measure over the trajectories.

Theorem 1.1.6. *Let $\Psi : [0,T] \times X \to X$[11] be such that uniqueness for the continuity equation (1.6) (with $f \equiv 0$) holds and let η be a generalised Ψ-flow. Then:*

i) for μ-a.e. $x \in X$, the measures $\mathbb{E}(\eta|w(0) = x)$ are Dirac masses in $\Omega(X)$, and setting
$$\mathbb{E}(\eta|w(0) = x) = \delta_{U(\cdot,x)}, \quad U(\cdot,x) \in \Omega(X),$$
where here δ_x denotes the Dirac function in x, the map $U(t,x)$ is a Ψ-flow.

ii) Any other generalised Ψ-flow coincides with η. In particular U is the unique flow associated to Ψ in the sense that, if $\tilde{U}$ is another Ψ-flow, then
$$U(\cdot,x) = \tilde{U}(\cdot,x), \qquad \mu - a.e. x \in X.$$

The proof of this theorem is uniquely based on Proposition 1.1.1 and uniqueness of solutions of the corresponding continuity equation, which in turn could follow as a corollary of the next stability theorem. This estimate was obtained in [9] by L. Ambrosio, E. Bruè and D. Trevisan and, as the authors remark, still holds for generalised flows (below we state it in this situation).

Theorem 1.1.7 (Stability estimate)**.** *Let $p \in (1,\infty)$ and $\eta, \bar{\eta}$ be generalised flows with compressibility constants $L, \bar{L}$ and associated to the vector fields*
$$\Psi \in L^1((0,T); W^{1,p}(X,\mu;X)) \qquad \bar{\Psi} \in L^1((0,T); L^1_\mu(X;X))$$
with $\|\Psi - \bar{\Psi}\|\|_{L^1_\mu((0,T)\times X;X)} < 1$, then
$$\int_X \int_{Y_x} \int_{\bar{Y}_x} \|y(t) - \bar{y}(t)\|_X \wedge 1 d\bar{\eta_x} d\eta_x d\mu \leq \frac{C}{|\log \|\Psi - \bar{\Psi}\|_{W^{1,p}_\mu}|}, \quad \forall\, t \in [0,T] \tag{1.9}$$

[11] This theorem holds for vector fields non necessarily valued in the Cameron-Martin space H.

where η_x *and* $\bar{\eta_x}$ *denote the generalised* Ψ *and* $\bar{\Psi}$*-flows concentrated over* $Y_x := \{y \in C((0,T);X) : y(t) = x + \int_0^t \Psi(y(s))ds\}$ *and* $\bar{Y}_x := \{\bar{y} \in C((0,T);X) : \bar{y}(t) = x + \int_0^t \bar{\Psi}(\bar{y}(s))ds\}$.

Corollary 1.1.1 (Uniqueness)**.** *Under the assumptions of the stability Theorem 1.1.7, there exists at most one* η_x *generalised* Ψ*-flow starting from* $x \in X$ *at time zero.*

Remark 1.1.4. As a by product of the uniqueness result we would get, following [10], that the law of the constructed process is a Dirac measure over its trajectories. This would imply, on one hand, that the respective flows are undistinguishable and in this sense unique, on the other hand that the solution is in fact deterministic for every initial data in the support of the measure. Meaning, in comparison with (1.5), that the following strongest expression would hold

$$U(t,x) = x + \int_0^t \Psi(U(s,x))ds, \quad \mu - a.e.\ x \in X,\ \forall t \in \mathbb{R}. \tag{1.10}$$

1.2 Models from hydrodynamics

My main motivation is the study of statistical solutions of the Euler equations for initial data in the support of invariant measures, usually for very irregular initial data. This study is based on the works [66, 34, 33] concerning the following models. Finally, we conclude this thesis by considering a stochastic perturbation of the Navier-Stokes equations on the two-dimensional torus, this is the content of Chapter 5.

- **the two-dimensional averaged-Euler equations on the torus** $\mathbb{T}^2 \simeq [0,2\pi]^2$**.** For an incompressible non-viscous fluid they are the following

$$\frac{\partial Au}{\partial t} + (u \cdot \nabla)Au + (\nabla u)^T \cdot Au = -\nabla p, \qquad \nabla \cdot u = 0, \tag{1.11}$$

  where $A = (1 - a^2\Delta)^s$ for a a real parameter and s a positive number. The mean velocity of the flow is denoted by $u : \mathbb{T}^2 \to \mathbb{T}^2$ and the pressure by $p : \mathbb{T}^2 \to \mathbb{T}^2$. These equations were first introduced by the authors of [44] in order to consider a modification of the Euler model such that nonlinear effects at small scales of the motion are negligible; therefore the dynamics remains turbulent, but non dissipative.

  Global existence and uniqueness for solutions (non probabilistic) of the two-dimensional averaged-Euler equation are known both in $\mathbb{R}^2$ and in a bounded domain for initial velocities in H^3, see respectively V. Busuioc [22] and S. Shkoller [63]. In the latter reference the classical PDE problem is transformed into a geometric one, considering geodesics in the (infinite-dimensional) group of volume-preserving diffeomorphisms. Averaged-Euler equations are indeed known to describe the velocities of geodesics on this group endowed with the H^1 metric. For further results concerning these equations we cite [23] and references therein.

This is the subject of the work [66] by which Chapter 2. below is inspired.

- **The incompressible non-viscous Euler equations on $\mathbb{R}^2$.**

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \qquad \nabla \cdot u = 0 \tag{1.12}$$

where $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ denotes the time dependent velocity field and $p : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ denotes the pressure. The first equation is Newton's second law (the acceleration is proportional to the pressure) and the second equation is the incompressibility condition.

Many different results are known about existence of solutions for the Euler equations: Lichtenstein in 1925 proved existence of local classical solutions [50]; Judovic in 1963 took advantage of the infinitely many conserved quantities of the equations to construct global weak solutions [46]; Arnold's representation of solutions as geodesics in the group of measure preserving diffeomorphism of L^2 [12] dates back to 1966 and measure-valued solutions were first defined by Majda and DiPerna [36] in 1986. Also for weak solutions of the Euler equation, uniqueness is an issue. Indeed, both in 2D and 3D it is possible to construct non-trivial weak solutions of the Euler equation with compact support in space and time which imply non-uniqueness, see [61, 64].

This is the subject of the work [34] by which Chapter 3. below is inspired.

- **A modification of the Euler equations on $\mathbb{R}^2$.** The modification concerns the pressure contribution, namely

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla)\tilde{u} = -\nabla p + cxp, \qquad \mathrm{div}\tilde{u} = 0 \tag{1.13}$$

where the pressure $p : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ may depend on c and $c \in (0, 1)$ is a fixed parameter. After the change of variables

$$u(t, x) = \sigma^c(x)\tilde{u}(t, x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^2 \tag{1.14}$$

where $\sigma^c(x) = \frac{1}{2\pi} e^{-\frac{c|x|^2}{2}}$ denotes a Gaussian density in $\mathbb{R}^2$, the equation reads,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)(\rho^c u) = -\nabla(\sigma^c p), \qquad \mathrm{div}_{\rho^c} u = 0, \tag{1.15}$$

where $\rho^c(x) := (\sigma^c)^{-1}(x) = 2\pi e^{\frac{c|x|^2}{2}}$ and $\mathrm{div}_{\rho^c} u$ is defined by

$$\int_{\mathbb{R}^2} \mathrm{div}_{\rho^c} u f d\rho^c = -\int_{\mathbb{R}^2} u \cdot \nabla f d\rho^c, \qquad \forall f \in \mathcal{C}^1_c$$

(for simplicity, we use the notation $d\rho^c = \rho^c dx$). We assume that the initial condition for (1.15) is defined by $u_0 = \sigma^c \tilde{u}_0$, where $\tilde{u}_0$ is the initial data for (1.13), and that $\tilde{u}$ and u vanish sufficiently rapidly at infinity. As we will see below, this change of variables allows us to study the equations in $L^2_{\sigma^c}(\mathbb{R}^2)$, the space of real-valued functions that are square integrable with respect to the measure $\sigma^c dx$. Similarly to the Euler equations, its modification has an infinite number of conserved quantities. This fact allows us to show existence of weak solutions (in the classical sense) in the same spirit of Judovic's work for the Euler equations.

This is the subject of the work [33] which is the content of Chapter 4. below.

- **Stochastic Navier-Stokes equations on $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$.**

  They describe the motion of an incompressible viscous flow

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + \varepsilon \Delta u - \nabla p + \dot{B}_t, \qquad \nabla \cdot u = 0,$$

  where $\varepsilon > 0$ denotes the viscosity coefficient and $\dot{B}_t$ the perturbation given by the formal derivative of a normalized cylindric Brownian motion. The mean velocity of the flow is denoted by $u : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}^2$ and the pressure by $p : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$.

  Within the firsts to consider a stochastic version of the Navier-Stokes equations we recall A. Bensoussan [16]; M. Capiński and N. Cutland [26]; Z. Brzeźniak, M. Capiński and F. Flandoli [20]. For these equations, F. Flandoli studied dissipativity and invariant measures [42] and joint with D. Gatarek the existence of martingale and stationary solutions [43]. Ergodicity for the three-dimensional stochastic Navier-Stokes equations was studied by G. Da Prato and A. Debussche [60] cf. also [32]. Existence of global L^2-solutions was proved by R. Mikulevicius and B. L. Rozovskii [56].

  This is the subject of Chapter 5.

1.2.1 Main results

• **The two-dimensional averaged-Euler equations on the torus $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$.** I constructed Gaussian measures of Gibbsian-type for the averaged-Euler equation with respect to the "enstrophy",

$$S := \frac{1}{2} \int_{\mathbb{T}^2} (A \Delta \varphi)^2 dx,$$

here φ denotes the stream function corresponding to the vorticity formulation of (1.11), see Section 2.2. The enstrophy is one of the quantities conserved by the motion, thus the Gaussian probability measure whose covariance operator is given by this quantity is formally invariant under the averaged-Euler flow. Consider the probability measures on $\mathbb{C}$ defined for $\gamma \in \mathbb{R}^+$ by

$$d\mu_{\gamma,k}(z) = \frac{\gamma k^4 (1 + a^2 k^2)^{2s}}{2\pi} \exp\left\{ -\frac{1}{2} \gamma k^4 (1 + a^2 k^2)^{2s} |z|^2 \right\} dxdy$$

where $z = x + iy$. Then, we define these measures as follows:

$$d\mu_\gamma(\varphi) = \prod_{k>0} d\mu_{\gamma,k}(\omega_k),$$

where ω_k denotes the k-th order component of the Fourier expansion of φ in the orthonormal basis of $L^2([0,2\pi]^2)$ given by the eigenfunctions of the Laplacian operator with periodic boundary conditions; namely $e_k = e^{ik\cdot x}$ for all $k \in \mathbb{Z}^2$ and for all $x \in \mathbb{T}^2$, here $k \cdot x = k_1x_1 + k_2x_2$.

The triple $(H^{1-\alpha,s}, H^{2,s}, \mu_\gamma)$ is a complex abstract Wiener space with measurable norm $\|\cdot\|_{1-\alpha,s}$ for any $\alpha > -\frac{s}{s+1}$. We prove that these probability measures are infinitesimally invariant, namely we show $\delta_{\mu_\gamma} B = 0$ where with B we denote the averaged-Euler vector field, and that B is $L^p_{\mu_\gamma}$ integrable with respect to these. These facts are sufficient to prove existence of statistical solutions for which the measures μ_γ are invariant in the sense of the following theorem.

Theorem 1.2.1. *There exists a flow $U(t,\omega)$ defined on a probability space $(\Omega, \mathcal{F}, P_\gamma)$ with values in $H^{1-\alpha,s}$, $\alpha > 2$, $U(\cdot,\omega) \in C(\mathbb{R}; H^{1-\alpha,s})$, $\omega \in \Omega$ such that*

1.
$$U_k(t,\omega) = U_k(0,\omega) + \int_0^t B_k(U(s,\omega))ds, \quad P_\gamma - a.e.\, \omega \in \Omega,\ \forall t \in \mathbb{R}, \tag{1.16}$$

2. *and such that the measure μ_γ is invariant for the flow, in the sense that:*

$$\int f(U(t,\omega))dP_\gamma(\omega) = \int f d\mu_\gamma, \quad \forall t \in \mathbb{R}, \forall f \in \mathcal{D}. \tag{1.17}$$

Other conserved quantities, for example the energy E, may be used to the same purpose of defining (formally) invariant measures, say $\mu_{\gamma,E}$. On one hand, the averaged-Euler drift, B, is not square integrable with respect to the measures $\mu_{\gamma,E}$, thus it is not possible to construct invariant probabilistic solutions with respect to these measures. On the other hand, the energy is regular enough to define a measure (this time, not necessarily Gaussian) on its level sets (see Appendix A) and to construct, here, an averaged-Euler velocity. Namely, we have

Theorem 1.2.2. *Let $r > 0$ be such that $\rho(r) > 0$ (we denote by $\rho(r) = \frac{d(E*\mu_\gamma)}{dr}$); then there exists a Borel probability measure defined on $H^{1-\alpha,s}$, ν^r_γ, with support on $V_r = \{\varphi\,|E(\varphi) = r\}$ and such that*

$$\int g^*(\varphi)d\nu^r_\gamma = \frac{\rho_g(r)}{\rho(r)},$$

for any g^* redefinition [12] *of g.*

[12] See Definition A.1.4. of Appendix A.

We prove infinitesimally invariance:

Theorem 1.2.3.
$$\int < B^n, \nabla f >^*_{2,s} d\nu^r_\gamma = 0, \qquad \forall f \in \mathcal{D}$$
for any $< B^n, \nabla f >^*_{2,s}$ *redefinition of* $< B^n, \nabla f >_{2,s}$.

Existence of a flow on the level sets of the energy:

Theorem 1.2.4. *Let* $\alpha > 2$. *For all* $r > 0$ *such that* $\rho(r) > 0$, *there exists a flow* $U'(\cdot, \omega)$ *defined on a probability space* $(\Omega, \mathcal{F}, P^r_\gamma)$ *with values in* V_r, $U'(\cdot, \omega) \in C(\mathbb{R}; V_r)$, $\omega \in \Omega$ *such that:*

1. *for any* B^* *redefinition of* B,
$$U'_k(t, \omega) = U'_k(0, \omega) + \int_0^t B^*_k(U'(s, \omega))ds, \quad P^r_\gamma - a.e.\, \omega,\ \forall t \in \mathbb{R},$$

2. ν^r_γ *is invariant for the flow, in the sense that:*
$$\int f(U'(t, \omega))dP^r_\gamma(\omega) = \int f(\varphi)d\nu^r_\gamma(\varphi), \qquad \forall t \in \mathbb{R},\ \forall f \in \mathcal{D}.$$

Moreover, I show that the solutions are recurrent in the sense that, starting from any initial point in the support, the solution returns (a.e. and infinitely many times) in a neighborhood of the initial value.

Theorem 1.2.5. *Let* $\alpha > 2$ *and fix* $\varphi_0 \in V_r \subset H^{1-\alpha,s}$. *If* $\varepsilon > 0$ *is sufficiently small, then for* ν^r_γ*-a.e.* $\varphi \in V_r \subset H^{1-\alpha,s}$ *such that* $\|\varphi - \varphi_0\|_{1-\alpha,s} < \varepsilon$, *there exists a sequence* $\{t_n\} \uparrow \infty$ *such that the corresponding invariant flow starting from* φ, $U'_\varphi(t, \omega)$, *satisfies* $\mathbb{E}_{P^r_\gamma} \|U'_\varphi(t_n, \omega) - \varphi_0\|_{1-\alpha,s} < 2\varepsilon$.

• **The incompressible non-viscous Euler equations on $\mathbb{R}^2$.** Existence of probabilistic integral flows for the two-dimensional Euler equation on the torus $[0, 2\pi] \times [0, 2\pi]$ was studied in [3]. By simply rescaling we can consider these flows on the phase-space $[0, L] \times [0, L]$ and existence of Euler velocities on the plane is studied by considering the limit when the period L tends to infinity. Indeed, it is possible to formally define stochastic processes, $\{\Phi_L\}_{L\in\mathbb{N}^*}$, (with values in the Sobolev space H^β for $\beta < 1$) whose laws are the invariant measures constructed in [3] (after rescaling), that we denote by $\mu_{L,\gamma}$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and for each $L > 0$, we consider the stochastic process
$$\Phi_{L,R}(\omega, x) := \sum_{k>0} a^L_k(\omega) e^L_k(x),$$

where

$$a_k^L(\omega) := \chi_k(\omega)\sqrt{\frac{2}{\gamma}}\left(\frac{L}{2\pi k}\right)^2,$$

and $\{\chi_k\}_{k\in\mathbb{Z}^2}$ denotes a sequence of complex-valued i.i.d. Gaussian random variables. We show that these processes form a Cauchy sequence in $L^2(\Omega; H^\beta_{loc}(\mathbb{R}^2))$ and this fact implies that the measures $\mu_{L,\gamma}$ converge weakly to some measure μ_γ with respect to the topology of $H^\beta_{loc}(\mathbb{R}^2)$, which is also shown to be the support of μ_γ for values of β smaller than one. Namely,

Theorem 1.2.6. *The sequence $\{\Phi_L\}_{L\in\mathbb{N}^*}$ is a Cauchy sequence in $L^2(\Omega; H^\beta_{loc}(\mathbb{R}^2))$ for $\beta < 1$.*

and

Theorem 1.2.7. *Let $\beta < 1$, we have*

$$\mathrm{supp}(\mu_\gamma) = H^\beta_{loc}(\mathbb{R}^2).$$

We can prove existence of an integral flow.

Theorem 1.2.8. *Let $\beta < -1$. There exists a probability space $(\Omega, \mathcal{F}, P)$ and globally defined flow $U(\cdot,\omega) \in C(\mathbb{R}; H^\beta_{loc}(\mathbb{R}^2))$ for P- a.e. $\omega \in \Omega$, such that*

1.
$$U(t,\omega) = U(0,\omega) + \int_0^t B(U(s,\omega))ds, \qquad P - a.e.\,\omega,\ \forall\ t \in \mathbb{R},$$

2. *the measure μ_γ is invariant under the flow, in the sense that*
$$\int f(U(t,\omega))dP(\omega) = \int f(\varphi)d\mu_\gamma(\varphi), \qquad \forall f \in C_b,\ \ \forall\ t \in \mathbb{R}.$$

Last, we also show that the flow is P-almost everywhere continuous from $H^\beta_{loc}(\mathbb{R}^2)$ to $H^\beta_{loc}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$.

• **A modification of the Euler equations on** $\mathbb{R}^2$**.** The modification concerns the pressure term and it is performed with the intent of keeping a vorticity formulation in potential form after the change of variable $u = \frac{1}{2\pi}e^{-\frac{c|x|^2}{2}}\tilde{u}$, where u denotes the usual Euler velocity on the plane and $\tilde{u}$ the velocity of the modified Euler equation. This change of coordinates enables one to work in the space of square-integrable functions with respect to the Gaussian measure $\frac{1}{2\pi}e^{-\frac{c|x|^2}{2}}$ for which an orthonormal basis of eigenfunctions of the Ornstein-Uhlenbeck operator is known (Hermite's polynomials). Within this settings, the vorticity equation for the modified Euler model corresponds to the classical one with the

Laplacian replaced by the Ornstein-Uhlenbeck operator. The vorticity equation now reads

$$\frac{\partial}{\partial t}L^c\varphi = -(\nabla^\perp\varphi\cdot\nabla)L^c\varphi. \tag{1.18}$$

In particular, we observe that the quantity $L^c\varphi$ is conserved along the particle trajectories with velocity $\tilde{u}$, that we denote by Φ_t, that is

$$L^c\varphi(t,x) = L^c\varphi(0,\Phi_{-t}(x)), \qquad t\in\mathbb{R},\ x\in\mathbb{R}^2. \tag{1.19}$$

The L^p-norms of $L^c\varphi$ are conserved for all $p\in\{1,2,\dots,\infty\}$. By equation (1.19), we obtain weak solutions of the vorticy equation if we are able to solve the associated ODE for the particle trajectories

$$\begin{aligned}\frac{d}{dt}\Phi_t(x) &= \rho^c u(\Phi_t(x),t)\\ \Phi_0(x) &= x.\end{aligned} \tag{1.20}$$

For initial data ω_0 such that $\rho^c\omega_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ we prove the following

Theorem 1.2.9. *Given $\rho^c\omega_0\in L^1\cap L^\infty$, there exists $T>0$ such that equation (1.20) has a unique solution in $[-T,T]$ and $\rho^c\omega\in L^\infty([-T,T];L^1\cap L^\infty)$ is a weak solution for equation (1.18).*

We prove also that the supports of the measures $\mu_{\sigma^c,\gamma}$ are not only spaces of very irregular functionals, but that in fact contain regular functions. Namely, $L^p_{loc}(\mathbb{R}^2)\subset \operatorname{supp}(\mu_{\sigma^c,\gamma})$ for every $p\in(2,10/3)$. To prove this we use the so called "dispersive bound" for Hermite functions, firstly proved in dimension one by N. Burq, L. Thomann and N. Tzvetkov in [21] and extended to other dimensions by A. Poiret in his Ph.D. thesis [58].

Theorem 1.2.10. *Let $\varepsilon>0$ and $p\in(2,\frac{10}{3})$; then*

$$\operatorname{supp}_{\mu_{\sigma^c,\gamma}} = H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2)\cap L^p_{loc}(\mathbb{R}^2).$$

After studying the L^r-regularity of B^c, its derivatives and divergence with respect to the measures $\mu_{\sigma^c,\gamma}$, we show the existence of a quasi-invariant measure and, thank to Üstunel's result, existence of a unique flow follows.

Theorem 1.2.11. *Let $\beta=2$ and $\varepsilon>0$, then $B^c: H^{-\varepsilon}_{\sigma^c}\to H^2_{\sigma^c}$ is such that there exists an almost surely unique flow for B^c defined by*

$$U^c_t(\varphi) = \varphi + \int_0^t B^c(U^c_s(\varphi))ds \qquad \mu_{\sigma^c,\gamma} - a.e.\ \varphi\in H^{-\varepsilon}_{\sigma^c}\cap L^p_{loc}, \quad \forall t\in\mathbb{R}. \tag{1.21}$$

Moreover, the measure $\mu_{\sigma^c,\gamma}$ is quasi-invariant under U^B_t and

$$k_t(\varphi) = \exp\left(\int_0^t \operatorname{div}_{\mu_{\sigma^c,\gamma}} B^c(U^c_{-s}(\varphi))ds\right) \tag{1.22}$$

is the corresponding Radon-Nikodym density, defined by $k_t := \frac{dU_t^c * \mu_{\sigma^c,\gamma}}{d\mu_{\sigma^c,\gamma}}$. *We have* $k_t \in L^r_{\mu_{\sigma^c,\gamma}}$, *for all* $r \geq 1$.

Clearly, the statistical solution constructed is a good approximation (for a "small" modification) of the Euler flow on the plane. However, when varying the parameter modification (that is when it tends to zero) we cannot formally consider the limit of the constructed solution, since this probabilistic solution is defined a.e. with respect to Gaussian measures (also depending on the modification parameter) that, in particular, are all singular with respect to each other.

• **Asymptotics for the stochastic Navier-Stokes equations.**

We study the limit of a perturbed Navier-Stokes flow when the viscosity coefficient $\varepsilon > 0$ converges to zero. We already know from [3] that a stochastic Navier-Stokes flow ω^ε exists (Theorem 5.1.1 below) for μ-a.e. initial data $x \in H^\beta$ for $\beta < -1$. Here μ denotes the invariant probability measure constructed in [3]. Using Prohorov's theorem, we show the existence of a weak limit up to a subsequence. Indeed, if we denote by ν^ε the law of ω^ε on $C(\mathbb{R}^+; H^\beta)$, that is

$$\nu^\varepsilon(\Gamma) = \mathbb{P} \times \mu\left(\{(x,w)\ :\ \omega^\varepsilon(\cdot,x,w) \in \Gamma\}\right), \qquad \Gamma \in \mathcal{B}(C(\mathbb{R}^+; H^\beta))^{13},$$

we can extract a converging subsequence from $\{\nu^\varepsilon\}_{\varepsilon>0}$. This is the statement of the following

Theorem 1.2.12. *Let* $\beta < -3$. *The set* $\{\nu^\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(C(\mathbb{R}^+; H^\beta))$[14] *is precompact.*

The proof of this result relies, in particular, on the invariance of the probability measure μ under the stochastic flow ω^ε and the fact that the flow ω^ε is uniformly bounded, that is

Lemma 1.2.1. *For* $\beta < -3$, *we have* $\mathbb{E}_\mu \mathbb{E}_x \sup_{t\in[0,T]} \|\omega^\varepsilon\|_\beta \leq C(T)$ *uniformly in* ε.

[13] With $\mathcal{B}(C(\mathbb{R}^+; H^\beta))$ we denote all the Borelian sets of $C(\mathbb{R}^+; H^\beta)$.
[14] With $\mathcal{M}(C(\mathbb{R}^+; H^\beta))$ we denote all the measures over $C(\mathbb{R}^+; H^\beta)$.

Chapter 2

Invariant measures for the two-dimensional averaged-Euler equations

2.1 Introduction

The purpose of this paper is to built invariant measures for the averaged-Euler equations. The averaged-Euler equations were introduced in 1998 by D. D. Holm, J. E. Marsden and T. S. Ratiu in [44]. For an incompressible non-viscous fluid the equations are the following

$$\frac{\partial Au}{\partial t} + (u \cdot \nabla)Au + (\nabla u)^T \cdot Au = -\nabla p, \qquad \nabla \cdot u = 0,$$

where $A = (1 - a^2\Delta)^s$ for a a real parameter and s a positive number. The mean velocity of the flow is denoted by $u : \mathbb{R}^2 \to \mathbb{R}^2$ and the pressure by $p : \mathbb{R}^2 \to \mathbb{R}$. The authors of [44] consider a modification of the Euler equations such that non linear effects at small scales of the motion are negligible; therefore the dynamics remains turbulent, but non dissipative.

Global existence and uniqueness for solutions of the two-dimensional averaged-Euler equation are known both in $\mathbb{R}^2$ and in a bounded domain for initial velocities in H^3, see respectively V. Busuioc [22] and S. Shkoller [63]. In the latter reference the classical pde problem is transformed into a geometric one, considering geodesics in the (infinite-dimensional) group of volume-preserving diffeomorphisms. Averaged-Euler equations are indeed known to describe the velocities of geodesics on this group endowed with the H^1 metric. For further results concerning these equations we cite [23] and references therein.

For this system O. Bell, A. Chorin and W. Crutchfield pointed out in [15] that invariant Gibbs measures can be considered, as the equations conserve the energy and the enstrophy. In their perspective the invariant measures are used to perform numerical predictions of the dynamics. Invariant measures are of significant interest when employed to improve existing deterministic results, particularly when we deal with vector fields with low regularity, see for example [30], [10], [40]; and may be used, among others, to extend local to global existence results or to prove recurrence properties for a flow, see for example [69]. Also quasiinvariant measures may serve to the same purposes and besides they are sometimes supported on more regular spaces, see for example [68]. Finally we mention several works related to invariant measures: [3], [4], [27], [5] about the two-dimensional Euler equations; and [18], [24], [35] about other dispersive equations.

We consider an equivalent formulation, the vorticity formulation, of the averaged-Euler equations. On $\mathbb{R}^2$, for divergence free velocity fields, the "stream function" $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is defined by $u = \nabla^{\perp}\varphi := (-\partial_2\varphi, \partial_1\varphi)$. The vorticity formulation in terms of stream function is the following

$$\frac{\partial A\Delta\varphi}{\partial t} + (\nabla^{\perp}\varphi \cdot \nabla)A\Delta\varphi = 0. \tag{2.1}$$

Below we consider (2.1) on $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$ and with periodic boundary conditions.

In this paper we rigorously define for this system an infinite dimensional Gibbs measure with respect to the enstrophy, formally

$$\text{“}d\mu_\gamma \simeq \frac{1}{Z} e^{-\gamma \frac{Enstrophy}{2}} d\lambda\text{”},$$

where $d\lambda$ denotes "Lebesgue measure", Z a normalizing constant and γ is a positive parameter. We construct a flow for the averaged-Euler equations on the support of this measure. Namely, as previously done by A. B. Cruzeiro and S. Albeverio in [3] for the analogous case of the Euler equations, applying a combination of Prohorov and Skorohod's theorems to finite dimensional flow approximations, we can construct continuous flows for the averaged-Euler vector field on some probability space $(\Omega, \mathcal{F}, P_\gamma)$ with values in $H^{1-\alpha,s}$ for some $s > 0$ and $\alpha > \frac{2-s}{1+s}$, that is in a Sobolev space of negative order. Therefore these pointwise continuous flows belong to a distribution space. In particular we will have

$$U(t, \omega) = U(0, \omega) + \int_0^t B(U(s, \omega))ds, \qquad P_\gamma - a.e.\ \omega \in \Omega,\ \forall t \in \mathbb{R},$$

with μ_γ invariant under the flow. See Theorem 2.3.2 below.

We also consider, as previously done for the Euler equations by F. Cipriano in [27], the infinite-dimensional conditional measure defined on level sets of the energy. Comparing with the Euler case, there is no need here to define a renormalized energy, since in the averaged-Euler case the energy itself is square integrable with respect to μ_γ. This surface measure ν_γ^r, is also invariant and therefore pointwise continuous flows can be constructed on a probability space with values on the level sets of the energy, say V_r, for some positive r. We have

$$U'(t, \omega) = U'(0, \omega) + \int_0^t B^*(U'(s, \omega))ds, \qquad P_\gamma^r - a.e.\ \omega,\ \forall t \in \mathbb{R},$$

where B^* is any redefinition of B. Moreover ν_γ^r is invariant under the flow. See Theorem 2.5.2 below.

Finally since the Poincaré recurrence theorem holds, we have that the flow returns to a neighborhood of the initial state infinitely many times. The analogous for the Euler system was proved in [28] by A. Constantin and D. Levy.

In Section 2 we define the spaces of functions which are relevant in our work; we rewrite the vorticity formulation for the averaged-Euler equations as an infinite dimensional system of ordinary differential equations using the Fourier coefficients of the stream function. Here we also show that the energy and the enstrophy are conserved quantities. In Section 3 we rigorously define a Gibbs measure μ_γ and describe its support. Moreover we study the $L^p_{\mu_\gamma}$ regularity of the vector field and we show that it is divergence free with respect to μ_γ. Finally we construct a flow on a suitable probability space. In Section 4 we define the infinite dimensional conditional measure ν_γ^r with support on the level sets of the energy V_r. In Section 5 we show that ν_γ^r is invariant and prove existence of a flow defined on some probability space. Finally we show that the solution returns to a neighborhood of the initial state infinitely many times.

2.2 The averaged-Euler equations

Consider the operator $A = (1 - a^2\Delta)^s$ for a a real parameter and s a positive number; if s is not an integer, A is a pseudo-differential operator. The averaged-Euler equations for an incompressible non-viscous fluid on $\mathbb{R}^2$ are (c.f. [15], [44])

$$\frac{\partial Au}{\partial t} + (u \cdot \nabla)Au + (\nabla u)^T \cdot Au = -\nabla p, \qquad \nabla \cdot u = 0 \tag{2.2}$$

where $u : \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity of the flow and $p : \mathbb{R}^2 \to \mathbb{R}$ is the pressure. In what follows we denote $\nabla^\perp = (-\partial_2, \partial_1)$ where ∂_1, ∂_2 are the partial derivatives with respect to the first and second variable.

We have the following

Theorem 2.2.1. *A time dependent vector field u is a smooth solution of (2.2) if and only if there exists a smooth (real) function φ such that $u = \nabla^\perp \varphi$ and φ is a solution of the equation*

$$\frac{\partial A\Delta\varphi}{\partial t} + (\nabla^\perp\varphi \cdot \nabla)A\Delta\varphi = 0. \tag{2.3}$$

Proof. Taking the "curl" of (2.2),

$$\frac{\partial A\nabla^\perp \cdot u}{\partial t} + \nabla^\perp \cdot [(u \cdot \nabla)Au] + \nabla^\perp \cdot [(\nabla u)^T \cdot Au] = 0,$$

we get

$$\frac{\partial A\nabla^\perp \cdot u}{\partial t} + (u \cdot \nabla)A\nabla^\perp \cdot u = 0.$$

From the condition $\nabla \cdot u = 0$ we know that exists a real-valued function φ, called the stream function, such that $u = \nabla^\perp \varphi$; thus sufficiency is proved. To prove necessity let f be defined by

$$f = -\frac{\partial A \nabla^\perp \varphi}{\partial t} - (\nabla^\perp \varphi \cdot \nabla) A \nabla^\perp \varphi - (\nabla \nabla^\perp \varphi)^T \cdot A \nabla^\perp \varphi.$$

Taking the "curl" we get

$$-\frac{\partial A \Delta \varphi}{\partial t} - (\nabla^\perp \varphi \cdot \nabla) A \Delta \varphi = \nabla^\perp \cdot f,$$

then $\nabla^\perp \cdot f = 0$ by assumption and thus there exists a scalar function p such that $f = \nabla p$. The proof is performed in detail in [4] for the analogous case of the Euler system. □

Our general settings will be similar to the ones in [27, 3, 4] where the case of Euler equation is studied. We will consider our equations on the two-dimensional torus $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$ and with periodic boundary conditions, that is

$$\varphi(0, y, t) = \varphi(2\pi, y, t) \text{ and } \varphi(x, 0, t) = \varphi(x, 2\pi, t), \qquad \forall\ (x, y) \in \mathbb{T}^2, \forall\ t \in \mathbb{R}.$$

Remark 2.2.1. From the expression of the vorticity equation we remark that if $s = 0$ we are considering the Euler system.

2.2.1 Conserved quantities of the motion

The averaged-Euler equation is conservative, meaning that the "energy" (u, Au) is an invariant of the motion (the inner product is the one of $L^2(\mathbb{T}^2)$); also the "enstrophy" $(A\nabla^\perp \cdot u, A\nabla^\perp \cdot u)$ is a conserved quantity and we can write these quantities in terms of the stream function φ as

$$E = -\frac{1}{2} \int_{\mathbb{T}^2} \varphi A \Delta \varphi dx$$

and

$$S = \frac{1}{2} \int_{\mathbb{T}^2} (A \Delta \varphi)^2 dx.$$

We have in fact that

$$\begin{aligned} \frac{dE}{dt} &= -\left(\frac{\partial}{\partial t} A \Delta \varphi, \varphi \right) = \Big((\nabla^\perp \varphi \cdot \nabla) A \Delta \varphi, \varphi \Big) \\ &= \Big(\nabla^\perp \varphi \cdot \nabla \varphi, A \Delta \varphi \Big) = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} S &= \left(\frac{\partial A \Delta \varphi}{\partial t}, A \Delta \varphi \right) = -\Big((\nabla^\perp \varphi \cdot \nabla) A \Delta \varphi, A \Delta \varphi \Big) \\ &= -\Big(\nabla^\perp \varphi \cdot \nabla \varphi, A \Delta A \Delta \varphi \Big) = 0 \end{aligned}$$

since $\nabla^\perp \varphi \cdot \nabla \varphi = 0$.

2.2.2 Fourier expansion of the system

We want to write the averaged-Euler partial differential equation as an infinite dimensional ordinary differential equation by means of Fourier expansion series (c.f. [3, 4] for an analogous formulation of the Euler equation). We consider an orthonormal basis of $L^2(\mathbb{T}^2)$, $\{e_k(x)\}_{k\in\mathbb{Z}^2}$, defined by $e_k(x) = \frac{1}{2\pi}e^{ik\cdot x}$. These functions are eigenfunctions of the Laplace operator:

$$\Delta e_k(x) = -k^2 e_k(x), \qquad \forall k \in \mathbb{Z}^2.$$

Here $k\cdot x = k_1x_1 + k_2x_2$ for $k = (k_1, k_2) \in \mathbb{Z}^2$ and $x = (x_1, x_2) \in \mathbb{T}^2$ and $k^2 = k\cdot k$. We say that $k \in \mathbb{Z}^2$ is positive if $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$. The Sobolev spaces

$$\mathcal{H}^{2s+2}(\mathbb{T}^2) = \left\{ v : \mathbb{T}^2 \to \mathbb{R} : \int \sum_{|\alpha|\le 2s+2} |D^\alpha v(x)|^2 dx < +\infty \right\}$$

can be identified with the complex Hilbert spaces

$$H^{2,s} = \left\{ v = \sum_{k\in\mathbb{Z}^2} v_k e_k : \sum_{k>0} k^4(1+a^2k^2)^{2s}|v_k|^2 < +\infty \right\}$$

with inner product $< u, v >_{2,s} = \sum_{k>0} k^4(1+a^2k^2)^{2s} u_k\bar{v}_k$. For general $p \in \mathbb{R}$ we define

$$H^{p,s} = \left\{ v = \sum_{k\in\mathbb{Z}^2} v_k e_k : \sum_{k>0} k^{2p}(1+a^2k^2)^{ps}|v_k|^2 < +\infty \right\}$$

with inner product $< u, v >_{p,s} = \sum_{k>0} k^{2p}(1+a^2k^2)^{ps} u_k\bar{v}_k$.

Henceforth we write $\varphi(x,t) = \sum_{h>0}\omega_h(t)e_h(x)$ and we write the energy and the enstrophy as

$$E = \frac{1}{2}\|\varphi\|^2_{1,s}$$

and

$$S = \frac{1}{2}\|\varphi\|^2_{2,s}.$$

Set $h'^{\perp} = (-h'_2, h'_1)$ for $h' \in \mathbb{Z}^2$; we have

$$\begin{aligned}\nabla^\perp\varphi\cdot\nabla A\Delta\varphi &= -\frac{1}{2\pi}\sum_{\substack{h>0,\\ h'>0,\\ h'\neq h}} \omega_h\omega_{h'}(h\cdot h'^{\perp})h'^2(1+a^2h'^2)^s e_{h+h'}(x)\\ &= -\frac{1}{2\pi}\sum_{\substack{h>0,\\ k>0,\\ h'+h=k}} \omega_h\omega_{h'}(h\cdot h'^{\perp})h'^2(1+a^2h'^2)^s e_k(x).\end{aligned}$$

Hence equation (2.3) holds if and only if

$$-\sum_{k>0}\left[k^2(1+a^2k^2)^s\frac{d\omega_k}{dt}+\frac{1}{2\pi}\sum_{\substack{h+h'=k,\\ h>0}}\omega_h\omega_{h'}(h\cdot h'^{\perp})h'^2(1+a^2h'^2)^s\right]e_k(x)=0,$$

meaning that (2.3) can be written as an infinite dimensional ODE as follows:

$$\begin{aligned}2\pi k^2(1+a^2k^2)^s\frac{d\omega_k}{dt}&=-\sum_{\substack{h+h'=k,\\ h>0}}(h\cdot h'^{\perp})h'^2(1+a^2h'^2)^s\omega_h\omega_{h'}\\ &=\frac{1}{2}\sum_{\substack{h+h'=k,\\ h>0}}(h\cdot h'^{\perp})[h^2-h'^2](1+a^2h'^2)^s\omega_h\omega_{h'},\quad \forall k>0.\end{aligned}\tag{2.4}$$

From $h'^{\perp}\cdot h=-h'\cdot h^{\perp}$ we obtain the following form of the averaged-Euler equations:

$$\frac{d\omega_k}{dt}=B_k(\varphi),\qquad \forall k>0,$$

where the vector field B is defined by

$$B(\varphi)=\sum_k B_k(\varphi)e_k,\tag{2.5}$$

with

$$B_k(\varphi)=\frac{1}{2\pi}\sum_{h>0}\left[\frac{1}{k^2}(h^{\perp}\cdot k)(h\cdot k)-\frac{1}{2}(h^{\perp}\cdot k)\right]\frac{(1+a^2(k-h)^2)^s}{(1+a^2k^2)^s}\omega_h\omega_{k-h}.\tag{2.6}$$

2.3 Gaussian invariant measures for the averaged-Euler equations

The purpose of this section is to construct an infinite dimensional Wiener measure μ_γ defined on some suitable $H^{p,s}$ space which is invariant for the averaged-Euler equation. We study the $L^p_{\mu_\gamma}$ regularity of the averaged-Euler vector field B, which is μ_γ-divergence free. This latter property will unable us to construct a flow associated with B. We proceed as in [3] where the Euler equation is considered.

Consider the probability measures on $\mathbb{C}$ defined for $\gamma\in\mathbb{R}^+$ by

$$d\mu_{\gamma,k}(z)=\frac{\gamma k^4(1+a^2k^2)^{2s}}{2\pi}\exp\left\{-\frac{1}{2}\gamma k^4(1+a^2k^2)^{2s}|z|^2\right\}dxdy\tag{2.7}$$

where $z = x + iy$. Then

$$d\mu_\gamma(\varphi) = \prod_{k>0} d\mu_{\gamma,k}(\omega_k) \tag{2.8}$$

is a measure with support in $H^{1-\alpha,s}$ for any $\alpha > -\frac{s}{s+1}$; indeed,

$$\begin{aligned}\int \|\varphi\|^2_{1-\alpha,s} d\mu_\gamma(\varphi) &= \sum_{k>0} k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)} \prod_{h>0} \int |\omega_k|^2 d\mu_{\gamma,h}(\omega_h) \\ &= \frac{2}{\gamma} \sum_{k>0} \frac{1}{k^{2(1+\alpha)}(1+a^2k^2)^{s(1+\alpha)}} < +\infty, \qquad \forall \alpha > -\frac{s}{s+1}.\end{aligned}$$

Proposition 2.3.1. $(H^{1-\alpha,s}, H^{2,s}, \mu_\gamma)$ *is a complex abstract Wiener space with measurable norm* $\|\cdot\|_{1-\alpha,s}$ *for any* $\alpha > -\frac{s}{s+1}$.

Proof. Consider the operator $\Gamma : H^{2,s} \to H^{2,s}$ defined by

$$\Gamma e_k = \frac{1}{|k|^{1+\alpha}(1+a^2k^2)^{(1+\alpha)s/2}} e_k,$$

which is a Hilbert-Schmidt operator since

$$\|\Gamma\|^2_{H.S.} = \sum_{k>0} \frac{1}{k^{2(1+\alpha)}(1+a^2k^2)^{(1+\alpha)s}} < +\infty, \qquad \forall \alpha > -\frac{s}{s+1}.$$

Let now $u = \sum_k u_k e_k$ in $H^{2,s}$, then

$$\|\Gamma u\|^2_{2,s} = \sum_k k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}|u_k|^2 = \|u\|^2_{1-\alpha,s}.$$

Because Γ is a Hilbert-Schimdt operator such that $\|\Gamma u\|^2_{2,s} = \|u\|^2_{1-\alpha,s}$ we can say that $\|\cdot\|_{1-\alpha,s}$ is a measurable norm in the sense of Gross (that is for every $\varepsilon > 0$ there exists $P_0 \in \mathcal{F}$, where $\mathcal{F}$ is the partially ordered set of finite dimensional orthogonal projection P of the space $H^{2,s}$, such that $\mu_\gamma\{\|Pu\|_{1-\alpha,s} > \varepsilon\} < \varepsilon$, $\forall P \perp P_0 \in \mathcal{F}$). On the other hand $H^{1-\alpha,s}$ is the closure of $H^{2,s}$ with respect to the norm $\|\cdot\|_{1-\alpha,s}$, that is $(H^{1-\alpha,s}, H^{2,s}, \mu_\gamma)$ is a complex abstract Wiener space. Indeed μ_γ is a Wiener measure on $H^{1-\alpha,s}$, namely

$$\int e^{i\gamma l(u)} d\mu_\gamma(u) = e^{-\frac{1}{2}\gamma\|l\|^2_{2,s}}, \qquad \forall l \in (H^{1-\alpha,s})' \subset H^{2,s}.$$

□

In particular $\mathbb{E}_{\mu_{L,\gamma}}(u_k) = 0$, $\mathbb{E}_{\mu_{L,\gamma}}(u_k \bar{u}_{k'}) = \frac{2\delta_{k,k'}}{\gamma k^4(1+a^2k^2)^{2s}}$ and $\mathbb{E}_{\mu_{L,\gamma}}(|u_k|^{2p}) = \frac{2^p p!}{\gamma^p k^{4p}(1+a^2k^2)^{2sp}}$ for $p \geq 1$. For further studies on abstract Wiener spaces and results similar to Proposition 2.3.1 see [49].

2.3.1 Regularity of the vector field

We are looking for solutions of

$$\frac{d\omega_k}{dt} = B_k(\varphi), \qquad \forall k > 0$$

that belong to $H^{1-\alpha,s}$ for all $t > 0$. Let us prove that $B : H^{1-\alpha,s} \to H^{1-\alpha,s}$, where B is defined in (2.5). We can consider finite dimensional approximations of B, namely

$$B^n(\varphi) = \sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} B^n_k(\varphi)e_k(x), \tag{2.9}$$

that are vector fields on $\mathbb{C}^d$, $d = d(n)$, and write

$$B^n_k(\varphi) = \sum_{h^2\le n} \alpha_{h,k}\omega_h\omega_{k-h} \tag{2.10}$$

where $\alpha_i \in \mathbb{Z}^2$ for $i \in \{1, \dots, d\}$ and

$$\alpha_{h,k} = \frac{1}{2\pi}\left[\frac{1}{k^2}(h^\perp \cdot k)(h\cdot k) - \frac{1}{2}(h^\perp \cdot k)\right]\frac{(1 + a^2(k-h)^2)^s}{(1 + a^2k^2)^s}.$$

Proposition 2.3.2. *The vector field $B \in L^p_{\mu_\gamma}(H^{1-\alpha,s}; H^{1-\alpha,s})$ for all $\alpha > \frac{2-s}{1+s}$ and $p \ge 1$.*

Proof. It is sufficient to prove that $B \in L^{2p}_{\mu_\gamma}(H^{1-\alpha,s}; H^{1-\alpha,s})$ for all $\alpha > \frac{2-s}{1+s}$ with p odd.

$$\begin{aligned}
\mathbb{E}_{\mu_{L,\gamma}}\|B(\varphi)\|^{2p}_{1-\alpha,s} &= \mathbb{E}_{\mu_{L,\gamma}}\left(\sum_k k^{2(1-\alpha)}(1 + a^2k^2)^{(1-\alpha)s}|B_k(\varphi)|^2\right)^p \\
&= \mathbb{E}_{\mu_{L,\gamma}}\sum_{k_1,\dots,k_p}\prod_{i=1}^p k_i^{2(1-\alpha)}(1 + a^2k_i^2)^{(1-\alpha)s}|B_{k_i}(\varphi)|^2 \\
&\le \sum_{k_1,\dots,k_p}\prod_{i=1}^p k_i^{2(1-\alpha)}(1 + a^2k_i^2)^{(1-\alpha)s}\left(\mathbb{E}_{\mu_{L,\gamma}}|B_{k_i}(\varphi)|^{2p}\right)^{1/p} \\
&= \left[\sum_k k^{2(1-\alpha)}(1 + a^2k^2)^{(1-\alpha)s}\left(\mathbb{E}_{\mu_{L,\gamma}}|B_k(\varphi)|^{2p}\right)^{1/p}\right]^p.
\end{aligned}$$

From $B_k(\varphi) = \sum_h \alpha_{h,k}\omega_h\omega_{k-h}$ we get that

$$\begin{aligned}
\mathbb{E}_{\mu_{L,\gamma}}|B_k(\varphi)|^{2p} &= \mathbb{E}_{\mu_{L,\gamma}}\left(\sum_{\substack{h_1,\dots h_p\\ h'_1,\dots h'_p}}\prod_{i=1}^p \alpha_{h_i,k}\alpha_{h'_i,k}\omega_{h_i}\omega_{k-h_i}\bar\omega_{h'_i}\bar\omega_{k-h'_i}\right) \\
&\le \left[\sum_{h,h'}\alpha_{h,k}\alpha_{h',k}(\mathbb{E}_{\mu_{L,\gamma}}(\omega_h\omega_{k-h}\bar\omega_{h'}\bar\omega_{k-h'})^p)^{1/p}\right]^p.
\end{aligned}$$

Observe that, if $h \neq h'$ or $h \neq k-h'$,

$$\mathbb{E}_{\mu_{L,\gamma}}(\omega_h \omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'})^p = \mathbb{E}_{\mu_{L,\gamma}}(\omega_h)^p \mathbb{E}_{\mu_{L,\gamma}}(\omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'})^p$$

and for p odd $\mathbb{E}_{\mu_{L,\gamma}}(\omega_h)^p = 0$. Hence we have

$$\begin{aligned}
&\sum_{h,h'\neq 0,k} \alpha_{h,k}\alpha_{h',k}(\mathbb{E}_{\mu_{L,\gamma}}(\omega_h \omega_{k-h} \bar{\omega}_{h'} \bar{\omega}_{k-h'})^p)^{1/p} \\
&\qquad = \sum_{h,h'\neq 0,k} (\delta_{h,k-h'}\alpha_{h,k}\alpha_{h',k} + \delta_{h,h'}\alpha_{h,k}\alpha_{h',k})(\mathbb{E}_{\mu_{L,\gamma}}(|\omega_h|^{2p}|\omega_{k-h}|^{2p}))^{1/p} \\
&\qquad \leq 2 \sum_{h\neq 0,k} |\alpha_{h,k}|^2 (\mathbb{E}_{\mu_{L,\gamma}}|\omega_h|^{2p})^{1/p}(\mathbb{E}_{\mu_{L,\gamma}}|\omega_{k-h}|^{2p})^{1/p} \\
&\qquad = 2 \sum_{h\neq 0,k} |\alpha_{h,k}|^2 \frac{(2^p p!)^{1/p}}{\gamma h^4(1+a^2h^2)^{2s}} \frac{(2^p p!)^{1/p}}{\gamma (k-h)^4(1+a^2(k-h)^2)^{2s}} \\
&\qquad \leq c(p,\gamma)\frac{1}{(1+a^2k^2)^{2s}} \sum_{h\neq 0,k}\left[\frac{1}{4h^2(k-h)^2(1+a^2h^2)^{2s}}\right]
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_k k^{2(1-\alpha)}(1+a^2k^2)^{(1-\alpha)s}\left(\mathbb{E}_{\mu_{L,\gamma}}|B_k(\varphi)|^{2p}\right)^{1/p} \\
&\qquad \leq c(p,\gamma)\sum_k k^{2(1-\alpha)}(1+a^2k^2)^{-s(1+\alpha)} \sum_{h\neq 0,k}\left[\frac{1}{4h^2(k-h)^2(1+a^2h^2)^{2s}}\right]
\end{aligned}$$

where the series converge for $\alpha > \frac{2-s}{1+s}$. □

Corollary 2.3.1. *The convergence* $\lim_{n\to+\infty} B^n = B$ *holds in* $L^2_{\mu_\gamma}(H^{1-\alpha,s}; H^{1-\alpha,s})$.

Proof. To show this statement observe that

$$\mathbb{E}_{\mu_{L,\gamma}}(\|B^n(\varphi)-B(\varphi)\|^2_{1-\alpha,s}) = \sum_{k>0} k^{2(1-\alpha)}(1+a^2k^2)^{(1-\alpha)s}\mathbb{E}_{\mu_{L,\gamma}}(|B^n_k(\varphi)-B_k(\varphi)|^2) \leq \varepsilon$$

for $\alpha > \frac{2-s}{1+s}$ and n sufficiently big. In fact $\mathbb{E}_{\mu_{L,\gamma}}(|B^n_k(\varphi)-B_k(\varphi)|^2)$ is infinitesimal for n sufficiently big, since $\lim_{n\to+\infty} B^n_k(\varphi) = B_k(\varphi)$ for a.e $\varphi \in H^{1-\alpha,s}$ and B^n_k is a Cauchy sequence in $L^2_{\mu_\gamma}(H^{1-\alpha,s};\mathbb{C})$, that is for $0<n<m$

$$\begin{aligned}
\mathbb{E}_{\mu_{L,\gamma}}|B^n_k(\varphi)-B^m_k(\varphi)|^2 &= \sum_{\substack{n\leq h^2\leq m\\ n\leq h'^2\leq m}} \alpha_{h,h'}\alpha_{h',k}\mathbb{E}_{\mu_{L,\gamma}}(\omega_h\omega_{k-h}\bar{\omega}_{h'}\bar{\omega}_{k-h'}) \\
&= \frac{4}{\gamma^2}\sum_{\substack{n\leq h^2\leq m\\ n\leq h'^2\leq m}} \alpha_{h,h'}\alpha_{h',k}\frac{(\delta_{h,h'}+\delta_{h,k-h'})}{h^4(1+a^2h^2)^{2s}(k-h)^4(1+a^2(k-h)^2)^{2s}} \\
&\leq \frac{8}{\gamma^2}\sum_{n\leq h^2\leq m}\frac{|\alpha_{h,k}|^2}{h^4(1+a^2h^2)^{2s}(k-h)^4(1+a^2(k-h)^2)^{2s}} < +\infty
\end{aligned}$$

as we saw above. □

We shall consider the gradient operator in the sense of Malliavin calculus (c.f. [53]), that is, for $\psi : H^{1-\alpha,s} \to X$ where X is a Banach space, $\nabla\psi(u)$ is defined, for $u \in H^{1-\alpha,s}$, by

$$\nabla\psi(u)(v) = D_v\psi(u) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\psi(u+\varepsilon v)-\psi(u)], \quad v \in H^{2,s}$$

where the limit is taken μ_γ-a.e. in $H^{1-\alpha,s}$. The second derivative is defined by iteration of the first, that is $\nabla^2\psi(u)(v,w) = D_vD_w\psi(u)$ for $u \in H^{1-\alpha,s}$ and $v,w \in H^{2,s}$, etc. Observe that the successive gradients $\nabla^r\psi(u)$ belong to $H^{2,s}_{\otimes r}$ for $r \geq 1$. On the symmetric tensorial product $H^{2,s}_{\otimes r} = H^{2,s}\otimes\cdots\otimes H^{2,s}$ (r times) we consider the Hilbert-Schmidt norm.

For example, we can check that

$$D_{e_j}B(\varphi) = \sum_{k>0}(\alpha_{j,k}+\alpha_{k-j,k})\omega_{k-j}e_k \tag{2.11}$$

and

$$D_{e_i}D_{e_j}B(\varphi) = \sum_{\substack{k=i+j,\\ k>0}}(\alpha_{j,k}+\alpha_{i,k})e_k. \tag{2.12}$$

Hence given $\{\hat{e}_k\}_{k\in\mathbb{Z}^2}$ an orthonormal basis of $H^{2,s}$, namely $\hat{e}_k = \frac{e_k}{k^2(1+a^2k^2)^s}$, the Hilbert-Schmidt norms of ∇B and $\nabla^2 B$ are respectively

$$\begin{aligned}\|\nabla B(\varphi)\|^2_{H.S.} &= \sum_j \|\nabla B(\varphi)(\hat{e}_j)\|^2_{1-\alpha,s}\\ &= \sum_{j,k}\frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}}(\alpha_{j,k}+\alpha_{k-j,k})^2|\omega_{k-j}|^2\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\|\nabla^2 B(\varphi)\|^2_{H.S.} &= \sum_{i,j}\|D_{\hat{e}_i}D_{\hat{e}_j}B(\varphi)\|^2_{1-\alpha,s}\\ &= \sum_{i,j}\frac{(i+j)^{2(1-\alpha)}(1+a^2(i+j)^2)^{(1-\alpha)s}}{i^4j^4(1+a^2i^2)^{2s}(1+a^2j^2)^{2s}}(\alpha_{j,i+j}+\alpha_{i,i+j})^2.\end{aligned} \tag{2.14}$$

Proposition 2.3.3. *For all $\alpha > \frac{2-s}{1+s}$ and $p \geq 1$, $\nabla B \in L^p_{\mu_\gamma}(H^{1-\alpha,s}; H.S.(H^{2,s}, H^{1-\alpha,s}))$ and $\nabla^2 B \in L^p_{\mu_\gamma}(H^{1-\alpha,s}; H.S.(H^{2,s}\otimes H^{2,s}, H^{1-\alpha,s}))$.*

Proof.

$$\begin{aligned}
\mathbb{E}_{\mu_{L,\gamma}}\|\nabla B(\varphi)\|^{2p}_{H.S.} &= \mathbb{E}_{\mu_{L,\gamma}}(\|\nabla B(\varphi)\|^2_{H.S.})^p \\
&= \mathbb{E}_{\mu_{L,\gamma}}\left[\sum_{j,k}\frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}}(\alpha_{j,k}+\alpha_{k-j,k})^2|\omega_{k-j}|^2\right]^p \\
&= \mathbb{E}_{\mu_{L,\gamma}}\sum_{\substack{k_1,\dots,k_p\\ j_1,\dots,j_p}}\prod_{i=1}^{p}\frac{k_i^{2(1-\alpha)}(1+a^2k_i^2)^{s(1-\alpha)}}{j_i^4(1+a^2j_i^2)^{2s}}(\alpha_{j_i,k_i}+\alpha_{k_i-j_i,k_i})^2|\omega_{k_i-j_i}|^2 \\
&\leq \sum_{\substack{k_1,\dots,k_p\\ j_1,\dots,j_p}}\prod_{i=1}^{p}\frac{k_i^{2(1-\alpha)}(1+a^2k_i^2)^{s(1-\alpha)}}{j_i^4(1+a^2j_i^2)^{2s}}(\alpha_{j_i,k_i}+\alpha_{k_i-j_i,k_i})^2(\mathbb{E}_{\mu_{L,\gamma}}|\omega_{k_i-j_i}|^{2p})^{1/p} \\
&= \left[\sum_{j,k}\frac{k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)}}{j^4(1+a^2j^2)^{2s}}(\alpha_{j,k}+\alpha_{k-j,k})^2\frac{(2^pp!)^{1/p}}{\gamma(k-j)^4(1+a^2(k-j)^2)^{2s}}\right]^p \\
&= \frac{(2^pp!)}{\gamma^p}C^p < +\infty
\end{aligned}$$

As we have shown in the proof of Proposition 2.3.2, the series above are convergent for every $\alpha > \frac{2-s}{1+s}$ and $p \geq 1$. For the second derivative of B, the statement follows straightforward from the fact that (2.14) converges for every $\alpha > \frac{2-s}{1+s}$ and $p \geq 1$. $\square$

2.3.2 The vector field is divergence free

Recall that on an abstract Wiener space (X, H, μ_γ), the divergence of a vector field $\Psi : X \to G$, $\Psi \in L^2_{\mu_\gamma}(X; G)$, where G is a Hilbert space, is defined by

$$\int \delta_{\mu_\gamma}\Psi \cdot f d\mu_\gamma = \int (\Psi, \nabla f)_G d\mu_\gamma, \quad \forall f \in \mathcal{D} \tag{2.15}$$

where $\mathcal{D}$ is the space of differentiable functions on X depending on a finite number of coordinates, that is $f(u) = f(u_{\alpha_1}, ..., u_{\alpha_d})$ where $d = d(n)$ and $(\cdot,\cdot)_G$ is the inner product of G.

Following [3], where the Euler equation is treated, we show that the averaged-Euler vector field is μ_γ-divergence free.

Theorem 2.3.1. *For $\alpha > \frac{2-s}{1+s}$ the vector field $B : H^{1-\alpha,s} \to H^{1-\alpha,s}$ defined above is divergence free with respect to the measure μ_γ, that is $\delta_{\mu_\gamma}B = 0$.*

Proof. Consider $d\mu^n_\gamma = \prod_{k\in\{\alpha_1,\dots,\alpha_d\}} d\mu_{\gamma,k}$ where $d = d(n)$ and denote by ρ^n_γ the density of this measure with respect to the Lebesgue measure. From the definition of divergence of a vector field and the fact that B^n converges to B in $L^2_{\mu_\gamma}(H^{1-\alpha,s}; H^{1-\alpha,s})$ when n goes to infinity, we have, for any $f \in \mathcal{D}$,

$$\begin{aligned}\int \delta_{\mu_\gamma} B \cdot f d\mu_\gamma(\varphi) &= \int < B, \nabla f >_{1-\alpha,s} d\mu_\gamma(\varphi) \\ &= \lim_n \int \sum_{k>0} k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)} B^n_k \overline{(\nabla f)_k} d\mu^n_\gamma(\varphi) = \lim_n \int < B^n \rho^n_\gamma, \nabla g >_{\mathbb{C}^d} dz\end{aligned}$$

where $g \in \mathcal{D}$ is defined by $g_k = k^{2(1-\alpha)}(1+a^2k^2)^{s(1-\alpha)} f_k$ for all $k \in \{\alpha_1, \ldots, \alpha_{d(n)}\}$. Therefore, for all $g \in \mathcal{D}$, we have

$$\lim_n \int \operatorname{div}(B^n \rho^n_\gamma) g dz = \lim_n \int \left[\operatorname{div} B^n + < B^n, \frac{\nabla \rho^n_\gamma}{\rho^n_\gamma} >_{\mathbb{C}^d} \right] g d\mu^n_\gamma(\varphi).$$

In particular,

$$\delta_{\mu^n_\gamma} B^n = \operatorname{div} B^n + < B^n, \frac{\nabla \rho^n_\gamma}{\rho^n_\gamma} >_{\mathbb{C}^d} = 0,$$

since on one hand, from the definition of B^n (equations (2.9) and (2.10))

$$\operatorname{div} B^n(\varphi) = \sum_{k \in \{\alpha_1, \ldots, \alpha_{d(n)}\}} D_{e_k} B^n_k(\varphi) = 0$$

and on the other hand,

$$< B^n, \frac{\nabla \rho^n_\gamma}{\rho^n_\gamma} >_{\mathbb{C}^d} = -\gamma < B^n(\varphi), \varphi >_{2,s} = 0$$

where the last equality holds by the conservation of the enstrophy. Therefore $\delta_{\mu^n_\gamma} B^n = 0$ for all $n \in \mathbb{N}$ and $\delta_{\mu_\gamma} B = 0$. □

Using the fact that B belongs to $L^2_{\mu_\gamma}$ and has divergence zero (with respect to the measure μ_γ), it is possible to construct a flow associated to B for which μ_γ is an invariant measure.

Lemma 2.3.1. *There exists a unique solution of* $\frac{dU^n(t,\varphi^n)}{dt} = B^n(U^n(t,\varphi^n))$, $U^n(0,\varphi^n) = \varphi^n(0)$ *which is defined for all times.*

Proof. For each $k > 0$, $B^n_k(\varphi^n)$ is a finite sum of quadratic terms,

$$B^n_k(\varphi^n) = \sum_{h^2 \leq n} \alpha_{h,k} \varphi^n_h \varphi^n_{k-h}.$$

Then existence of a unique solution follows from classical results on ordinary differential equations, while the conservation of the energy ensure that the solution is globally defined in time. □

Denote by $U^n(t,\varphi^n)$ the flow associated to B^n, that is $\varphi^n(0)\mapsto\varphi^n(t)$, and define the flow on $H^{1-\alpha,s}$ by

$$U^n(t,\varphi)=U^n(t,\varphi^n)+\Pi_n^\perp\varphi,$$

where $\Pi_n\varphi=\varphi^n$ stands for the orthogonal projection of u on the subspace spanned by $\{e_{\alpha_1},...,e_{\alpha_{d(n)}}\}$, then we have

$$\frac{dU^n(t,\varphi)}{dt}=B^n(U^n(t,\varphi)),\, U^n(0,\varphi)=\varphi(0),$$

in particular, $U^n(\cdot,\varphi)=\sum_k U_k^n(\cdot,\varphi)e_k$ where $U_k^n(\cdot,\varphi)\in C(\mathbb{R};\mathbb{C})$ for all $k>0$.

Theorem 2.3.2. *There exists a flow $U(t,\omega)$ defined on a probability space $(\Omega,\mathcal{F},P_\gamma)$ with values in $H^{1-\alpha,s}$, $\alpha>\frac{2-s}{1+s}$, $U(\cdot,\omega)\in C(\mathbb{R};H^{1-\alpha,s})$, $\omega\in\Omega$ such that*

1.
$$U_k(t,\omega)=U_k(0,\omega)+\int_0^t B_k(U(s,\omega))ds,\quad P_\gamma-a.e.\,\omega,\ \forall t\in\mathbb{R}, \tag{2.16}$$

2. and such that the measure μ_γ is invariant for the flow, in the sense that:
$$\int f(U(t,\omega))dP_\gamma(\omega)=\int fd\mu_\gamma,\quad \forall t\in\mathbb{R},\forall f\in\mathcal{D}. \tag{2.17}$$

Proof. The construction of such a flow can be found in [3] in the case of the two-dimensional Euler system. The same arguments apply in the case of the two-dimensional averaged-Euler equations. For $t\in\mathbb{R}^+$ consider $U_k^n(t,\varphi)$ as a stochastic process with law on $C(\mathbb{R}^+;\mathbb{C})$ defined by

$$\eta_k^n(\Gamma)=\mu_\gamma\{\varphi:U_k^n(\cdot,\varphi)\in\Gamma\},\qquad \Gamma\subset C(\mathbb{R}^+;\mathbb{C}).$$

Consider the sup-norm on $C(\mathbb{R}^+;\mathbb{C})$ and the weak topology on the space of measures over $C(\mathbb{R}^+;\mathbb{C})$. We have that:

1.
$$\eta_k^n(|y(0)|>R)\le\frac{1}{R^2}\mathbb{E}_{\mu_{L,\gamma}}(|\omega_k|^2)=\frac{2}{\gamma R^2k^4(1+a^2k^2)^{2s}}\to 0\qquad\text{when } R\to+\infty$$

2. for all $\rho>0$ and $T>0$
$$\begin{aligned}\eta_k^n\left(\sup_{\substack{0\le t\le t'\le T\\|t'-t|\le\delta}}|y(t)-y(t')|>\rho\right)&\le\frac{1}{\rho^2}\mathbb{E}_{\mu_{L,\gamma}}\left(\sup_{t,t'}|U_k^n(t,\varphi)-U_k^n(t',\varphi)|^2\right)\\&\le\frac{\delta}{\rho^2}\mathbb{E}_{\mu_{L,\gamma}}\int_0^T|B_k^n(U^n(s,\varphi))|^2ds\\&\le\frac{\delta T}{\rho^2}\mathbb{E}_{\mu_{L,\gamma}}|B_k^n|^2\le\frac{\delta TC}{\rho^2}\to 0\text{ when }\delta\to 0,\end{aligned}$$

in the last inequalities we used respectively that $U^n(t,\varphi)$ is a flow for B^n and that B^n has null divergence with respect to μ_γ for all n. By 1. and 2. we are under the assumptions of Prohorov's criterium; then there exists a subsequence of η_k^n (again denoted by η_k^n) that converges weakly to η_k. Remark that we can choose an arbitrary subsequence since k belongs to $\mathbb{Z}^2$ that is countable. Hence, by Skorohod's theorem, there exists a probability space $(\Omega,\mathcal{F},P_\gamma)$ and family of processes $U_k'^n(t,\omega)$, $U_k(t,\omega)$, $\omega\in\Omega$, having laws respectively η_k^n and η_k on $C(\mathbb{R}^+;\mathbb{C})$. Furthermore, $U_k'^n(\cdot,\omega)\to U_k(\cdot,\omega)$, P_γ-a.e. ω. Repeat for $t\in\mathbb{R}^+\mapsto U_k^n(-t,\varphi)$ to get the negative values of t. We now prove 2, take $f\in\mathcal{D}$,

$$\begin{aligned}\int f(U^n(t,\varphi))d\mu_\gamma &= \int d\mu_\gamma^{n,\perp}\int f(U^n(t,\varphi^n))d\mu_\gamma^n\\ &= \int d\mu_\gamma^{n,\perp}\int f d\mu_\gamma^n = \int f d\mu_\gamma, \qquad \forall t>0\end{aligned}$$

where $d\mu_\gamma^n=\prod_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}}d\mu_{\gamma,k}$ and $d\mu_\gamma^{n,\perp}=\prod_{k\notin\{\alpha_1,\dots,\alpha_{d(n)}\}}d\mu_{\gamma,k}$. On the other hand denoting by η^n the law of $U^n(\cdot,\varphi)$, we also have

$$\begin{aligned}\int f d\mu_\gamma &= \int f(U^n(t,\varphi))d\mu_\gamma = \int f(y(t))d\eta^n\\ &= \int f(U'^n(t,\omega))dP_\gamma \to \int f(U(t,\omega))dP_\gamma.\end{aligned}$$

Remark that $U(t,\omega)$ takes values in $H^{1-\alpha,s}$; in fact P_γ-a.e. $\omega\in\Omega$ we have

$$\int \|U(t,\omega)\|_{1-\alpha,s}^2 dP_\gamma = \int \|\varphi\|_{1-\alpha,s}^2 d\mu_\gamma < +\infty.$$

Finally we prove 1; we have

$$\begin{aligned}&\int\Big|\int_0^t [B_k^n(U'^n(s,\omega))-B_k(U(s,\omega))]ds\Big|dP_\gamma\\ &\quad\le \int\int_0^t |B_k^n(U'^n(s,\omega))-B_k(U'^n(s,\omega))|dsdP_\gamma\\ &\quad+\int\int_0^t |B_k(U'^n(s,\omega))-B_k(U(s,\omega))|dsdP_\gamma.\end{aligned}$$

The first integral converges through zero by the identification in law of $U'^n(t,\omega)$ and $U^n(t,\varphi)$, by the invariance of μ_γ under the flow $U^n(t,\varphi)$ and the fact that $B_k^n\to B_k$ in $L^2_{\mu_\gamma}$. The second integral converges towards zero by the dominated convergence theorem. Indeed $\{B_k(U'^n(s,\omega))\}_{n\in\mathbb{N}^*}$ is uniformly integrable on $[0,t]\times\Omega$,

$$\int\int_0^t |B_k(U'^n(s,\omega))|dsdP_\gamma = \int_0^t\int |B_k^n(\varphi)|d\mu_\gamma ds \le \int_0^t\int |B_k(U(s,\omega))|dP_\gamma ds \le Ct,$$

and $B_k(U_k'^n(s,\omega)) \to B_k(U_k(s,\omega))$ P_γ-a.e. ω for all $s \in [0,t]$ when n goes to infinity. The latter statement follows from the fact that $U_k'^n(\cdot,\omega) \to U_k(\cdot,\omega)$, P_γ-a.e. ω and that B_k^n converges uniformly to B_k (see Corollary 2.3.1) where

$$B_k(U'^n(\cdot,\omega)) = \sum_{h^2 \le n} \alpha_{h,k} U_h'^n(\cdot,\omega) U_{k-h}'^n(\cdot,\omega).$$

□

Remark 2.3.1. At this point we could ask ourselves about the possibility of considering the (Gibbs) measure associated to the energy, formally

$$\nu_\gamma \simeq \frac{1}{Z} e^{-\frac{\gamma E}{2}} \times \text{“}\textit{Lebesgue measure}\text{”},$$

where Z denotes a suitable normalizing constant, instead of the measure associated to the enstrophy μ_γ in (2.8). We can observe that the vector field B is not in L^2 with respect to ν_γ.

2.4 A surface measure

The energy of the averaged-Euler system belongs to the space $L^2_{\mu_\gamma}$. Therefore, as previously done in [27] for the Euler system (here a “renormalized” energy must be taken into account, because the energy is not square integrable with respect to the invariant measure), we consider the “surface” measure defined on the level sets of E, namely the conditional measure $\mu_\gamma(dx|E=r)$ for $r>0$. We want to take advantage of the fact that the energy E is also a conserved quantity of the motion in order to construct a flow for the averaged-Euler vector field with values on the level sets of E.

Remark 2.4.1. It is not possible to construct a flow on the level sets of E using the invariant measure; in fact $\mu_\gamma\{\varphi\,|E(\varphi)=r\}=0$.

We consider suitable Sobolev spaces on $(H^{1-\alpha,s}, H^{2,s}, \mu_\gamma)$: the space W_1^p of the maps $f : H^{1-\alpha,s} \to \mathbb{R}$ that belong to $L^p_{\mu_\gamma}(H^{1-\alpha,s};\mathbb{R})$ such that $\nabla f : H^{1-\alpha,s} \to H^{2,s}$, defined as $D_h f(x) =< \nabla f(x), h >_{2,s}$ for all $h \in H^{2,s}$ satisfy $\nabla f \in L^p_{\mu_\gamma}(H^{1-\alpha,s}; H^{2,s})$. More generally the space W_r^p, for every integer $r>1$, is the space of functions $f \in W_{r-1}^p$ such that $D_h f(x) \in W_{r-1}^p$ for all $h \in H^{2,s}$.

Proposition 2.4.1. *The energy E belongs to Sobolev spaces of all orders, that is $E \in W_\infty := \bigcap_{p,r} W_r^p$.*

Proof. First, we want to show that

$$(\mathbb{E}_{\mu_{L,\gamma}}|E(\varphi)|^{2m})^{1/m} < +\infty \quad \forall m = 2^{p-1} \text{ and } p \ge 2.$$

We have

$$(\mathbb{E}_{\mu_{L,\gamma}}|E(\varphi)|^{2m})^{1/m} \le \sum_k \left[\mathbb{E}_{\mu_{L,\gamma}}\left(k^2(1+a^2k^2)^s|\omega_k|^2\right)^{2m}\right]^{1/m}$$

and

$$\mathbb{E}_{\mu_{L,\gamma}}\left(k^2(1+a^2k^2)^s|\omega_k|^2\right)^{2m} \le c(p,\gamma)\frac{1}{k^{4m}(1+a^2k^2)^{2ms}};$$

thus

$$(\mathbb{E}_{\mu_{L,\gamma}}|E(\varphi)|^{2m})^{1/m} \le c(p,\gamma)\sum_k \frac{1}{k^4(1+a^2k^2)^{2s}}.$$

Now consider the linear functional $\nabla E(\varphi) : H^{2,s} \to \mathbb{R}$,

$$\nabla E(\varphi)(e_k) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(\varphi+\varepsilon e_k) - E(\varphi)\right) = 2k^2(1+a^2k^2)^s|\omega_k|,$$

and take $\hat{e}_k = \frac{e_k}{k^2(1+a^2k^2)^s}$ for all $k > 0$, orthonormal basis of $H^{2,s}$; then $\nabla E(\varphi)(\hat{e}_k) = 2|\omega_k|$ and

$$(\mathbb{E}_{\mu_{L,\gamma}}\|\nabla E(\varphi)\|_{2,s}^{2m})^{1/m} \le 4\sum_k(\mathbb{E}_{\mu_{L,\gamma}}|\omega_k|^{2m})^{1/m} \le c(p,\gamma)\sum_k\frac{1}{k^4(1+a^2k^2)^{2s}} < +\infty.$$

Finally observe that

$$(\mathbb{E}_{\mu_{L,\gamma}}\|\nabla^2 E(\varphi)\|_{H^{2,s}\otimes H^{2,s}}^{2m})^{1/m} = \sum_k \frac{1}{k^4(1+a^2k^2)^{2s}} < +\infty.$$

$\square$

Next proposition is proved in [27], following [52], in the case of the Euler system.

Proposition 2.4.2. *E is of maximal rank, that is $\|\nabla E\|_{2,s}^{-1} \in W_\infty$.*

Proof. We want to show that $\mathbb{E}_{\mu_{L,\gamma}}\|\nabla E(\varphi)\|_{2,s}^{-2p} < +\infty$ for all p. By Chebycheff inequality, for all $t > 0$

$$-e^{-\frac{t}{\varepsilon}}\mathbb{E}_{\mu_{L,\gamma}}\left(e^{-t\|\nabla E(\varphi)\|_{2,s}^{-2}}\right) \le \mu_\gamma\left\{\|\nabla E(\varphi)\|_{2,s}^2 \le \varepsilon\right\} \le e^{\frac{t}{\varepsilon}}\mathbb{E}_{\mu_{L,\gamma}}\left(e^{-t\|\nabla E(\varphi)\|_{2,s}^2}\right).$$

In particular

$$\mu_\gamma\left\{\|\nabla E(\varphi)\|_{2,s}^2 \le \varepsilon\right\} \ge -e^{-\frac{1}{\varepsilon}}\sum_{p\ge 0}\frac{(-1)^p}{p!}\mathbb{E}_{\mu_{L,\gamma}}(\|\nabla E(\varphi)\|_{2,s}^{-2p}),$$

meaning that $\mathbb{E}_{\mu_{L,\gamma}}\|\nabla E(\varphi)\|_{2,s}^{-2p}$ are finite for all p whenever $\mu_\gamma\left\{\|\nabla E(\varphi)\|_{2,s}^2 \leq \varepsilon\right\}$ is finite. We have

$$\begin{aligned}\mu_\gamma\left\{\|\nabla E(\varphi)\|_{2,s}^2 \leq \varepsilon\right\} &\leq e^{\frac{t}{\varepsilon}}\mathbb{E}_{\mu_{L,\gamma}}\left(e^{-t\|\nabla E(\varphi)\|_{2,s}^2}\right)\\ &= e^{\frac{t}{\varepsilon}}\prod_k\left(\frac{1}{1+\frac{8t}{\gamma k^4(1+a^2k^2)^{2s}}}\right)\\ &\leq e^{\frac{t}{\varepsilon}}\prod_{\{k\,:\,\gamma k^4(1+a^2k^2)^{2s}<\frac{8}{t}\}}\left(\frac{1}{1+t^2}\right)\\ &\leq \inf_t e^{\frac{t}{\varepsilon}}\left(\frac{1}{1+t^2}\right)^c < +\infty\end{aligned}$$

where $c = \#\{k\,:\,\gamma k^4(1+a^2k^2)^{2s} < \frac{8}{t}\}$. □

For $g \in W_\infty$, we shall denote by $\rho(r) = \frac{d(E*\mu_\gamma)}{dr}$ and by $\rho_g(r) = \frac{d(E*g\mu_\gamma)}{dr}$ respectively the C^∞ densities of $d(E*\mu_\gamma)$ and $d(E*g\mu_\gamma)$ with respect to the Lebesgue measure, see [53, 1]. As proved in [1], Propositions 2.4.1 and 2.4.2 ensure the existence of a conditional measure of μ_γ knowing that $E = r$ for $r > 0$.

Theorem 2.4.1. *Let $r > 0$ be such that $\rho(r) > 0$; then there exists a Borel probability measure defined on $H^{1-\alpha,s}$, ν_γ^r, with support on $V_r = \{\varphi\,|E(\varphi) = r\}$ and such that*

$$\int g^*(\varphi)d\nu_\gamma^r = \frac{\rho_g(r)}{\rho(r)},$$

for any g^*redefinition of g.

Proof. See [1]. □

Remark 2.4.2. Recall that, given a measurable function Φ with values in $\mathbb{R}^n$, we call a (p,r)-redefinition of Φ a function Φ^* such that $\Phi = \Phi^*$ a.s. and Φ^* is (p,r)-continuous (that is, if $\forall\varepsilon > 0$ it is possible to find an open set O_ε such that $c_{p,r}(O_\varepsilon) < \varepsilon$ and the restriction of Φ^* to O_ε^c is continuous). The capacity of the open set O is given by $c_{p,r}(O) = \inf\{\|u\|_{W_{2r}^p}; u \geq 0, u(x) \geq 1, \mu - \text{a.e. on } O\}$; O is said to be slim if $c_{p,r}(O) = 0$, for all $p, r \in \mathbb{N}$.

For all $\Phi \in W_\infty$ there exists a redefinition Φ^* and a sequence of open sets $\{O_n\}_{n\in\mathbb{N}}$ associated to this redefinition such that: $\bigcap_n O_n$ is slim, Φ^* is continuous on $(\bigcap_n O_n)^c$ and Φ^* and $\nabla^r\Phi^*$ are continuous on O_n^c for all $n, r \in \mathbb{N}$.

The proof of Theorem 2.4.1 is essentially based on the following considerations. For a fixed $\Phi \in W_\infty$ of maximal rank and non-degenerate and for $g \in W_\infty$ we consider the map

$$< \delta\Phi, g >: \xi \mapsto < \delta_\xi\Phi, g > := \rho_g(\xi)/\rho(\xi);$$

this map belongs to $C^\infty(O;\mathbb{R})$ where $O = \{\xi \in \mathrm{supp}(\Phi * \mu) \subset \mathbb{R}^n : \rho(\xi) > 0\}$. In particular the map

$$g \mapsto < \delta\Phi, g >$$

is a continuous linear functional from W_∞ to the space of functions C^∞ on O. If $S(O)$ is the Schwartz space of O and W' the dual of W_∞ (W' was accurately defined by Watanabe, see [53]) we can consider the dual map

$$\delta_*\Phi : S(O) \to W'$$

that associates linear functionals on W_∞ to distributions over $\mathbb{R}^n$ and such that

$$<< \delta_*\Phi, v >, g >=< v, < \delta\Phi, g >> \tag{2.18}$$

for every $v \in S(O)$ and $g \in W_\infty$. For further details see [53, 1].

We compute the second order moments of ν_γ^r. From [53] we know that $\rho_{\omega_k\bar{\omega}_{k'}} \in S(\mathbb{R})$ since $\omega_k\bar{\omega}_{k'} \in W_\infty$; then we have

$$\hat{\rho}_{\omega_k\bar{\omega}_{k'}}(r) = \int_{-\infty}^{+\infty} e^{ir\xi}\rho_{\omega_k\bar{\omega}_{k'}}(\xi)d\xi = \int_{H^{1-\alpha,s}} e^{irE(\varphi)}\omega_k\bar{\omega}_{k'}d\mu_\gamma.$$

If $k \neq k'$,

$$\begin{aligned}\hat{\rho}_{\omega_k\bar{\omega}_{k'}}(r) = \prod_{j\neq k,k'} \int_{\mathbb{C}} e^{\frac{ir}{2}j^2(1+a^2j^2)^s|\omega_j|^2} d\mu_{\gamma,j} \int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2}\omega_k d\mu_{\gamma,k}\\ \int_{\mathbb{C}} e^{\frac{ir}{2}k'^2(1+a^2k'^2)^s|\omega_k'|^2}\omega_{k'}d\mu_{\gamma,k'} = 0,\end{aligned}$$

if $k = k'$,

$$\hat{\rho}_{\omega_k\bar{\omega}_{k'}}(r) = \prod_{j\neq k} \int_{\mathbb{C}} e^{\frac{ir}{2}j^2(1+a^2j^2)^s|\omega_j|^2} d\mu_{\gamma,j} \int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2}|\omega_k|^2 d\mu_{\gamma,k},$$

where

$$\begin{aligned}&\int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2}|\omega_k|^2 d\mu_{\gamma,k}\\ &\quad = \frac{\gamma k^4(1+a^2k^2)^{2s}}{2\pi}\int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2-\frac{\gamma}{2}k^4(1+a^2k^2)^{2s}|\omega_k|^2}\omega_k\bar{\omega}_k dz\\ &\quad = \frac{1}{\gamma k^4(1+a^2k^2)^{2s} - irk^2(1+a^2k^2)^s}\int_{\mathbb{C}} e^{\frac{ir}{2}k^2(1+a^2k^2)^s|\omega_k|^2} d\mu_{\gamma,k}\end{aligned}$$

after complex by parts integration. Then

$$\hat{\rho}_{\omega_k\bar{\omega}_{k'}}(r) = \frac{1}{\gamma k^4(1+a^2k^2)^{2s} - irk^2(1+a^2k^2)^s}\int_{H^{1-\alpha,s}} e^{irE(\varphi)}d\mu_\gamma.$$

Hence

$$\begin{aligned}\rho_{\omega_k\bar{\omega}_{k'}}(\xi) &= \frac{1}{k^2(1+a^2k^2)^s}\int_{-\infty}^{+\infty} e^{-ir\xi}\frac{1}{\gamma k^2(1+a^2k^2)^s - ir}\int_{H^{1-\alpha,s}} e^{irE(\varphi)}d\mu_\gamma dr\\ &= \frac{1}{k^2(1+a^2k^2)^s}\int_{-\infty}^{+\infty} e^{-ir\xi}\frac{\hat{\rho}(\xi)}{\gamma k^2(1+a^2k^2)^s - ir}dr\end{aligned}$$

where

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-ir\xi}\frac{\hat{\rho}(\xi)}{\gamma k^2(1+a^2k^2)^s - ir}dr &= \left(\hat{\rho}(\xi)\frac{1}{\gamma k^2(1+a^2k^2)^s - ir}\right)^\vee\\ &= \rho(\xi) * \left(2\pi e^{-\gamma k^2(1+a^2k^2)^s y}\right),\end{aligned}$$

then

$$\rho_{\omega_k\bar{\omega}_{k'}}(\xi) = \frac{\pi}{k^2(1+a^2k^2)^s}\int_0^{+\infty}\rho(\xi+y)e^{-\gamma k^2(1+a^2k^2)^s y}dy.$$

We conclude that $\mathbb{E}_{\nu_\gamma^r}(\omega_k\bar{\omega}_{k'}) = \frac{\rho_{\omega_k\bar{\omega}_{k'}}(r)}{\rho(r)} = 0$ if $k \neq k'$ and $\mathbb{E}_{\nu_\gamma^r}(\omega_k\bar{\omega}_{k'}) = \frac{\pi}{k^2(1+a^2k^2)^s\rho(r)}\int_0^{+\infty}\rho(r+y)e^{-\gamma k^2(1+a^2k^2)^s y}dy$ if $k = k'$.

2.5 The invariant flow

2.5.1 Existence

Similar to [27], we show that the vector field B is divergence free with respect to the surface measure ν_γ^r. This will be fundamental for proving the existence of a flow on the level sets of the energy.

Theorem 2.5.1.

$$\int < B^n, \nabla f >^*_{2,s} d\nu_\gamma^r = 0, \qquad \forall f \in \mathcal{D}$$

for any $< B^n, \nabla f >^*_{2,s}$ *redefinition of* $< B^n, \nabla f >_{2,s}$.

Proof. Let $f \in \mathcal{D}$ and $v \in C_0^\infty(\mathbb{R})$ be arbitrary functions. We have,

$$\begin{aligned}\int_{\mathbb{R}} v(r)\rho(r)\int_{V_r} < B^n, \nabla f >^*_{2,s} d\nu_\gamma^r dr &= \int_{\mathbb{R}} v(r) d(E* < B^n, \nabla f >_{2,s} \mu_\gamma)\\ &= \int_{H^{1-\alpha,s}} < v(E(\varphi))B^n, \nabla f >_{2,s} d\mu_\gamma\\ &= \int_{H^{1-\alpha,s}} \delta_{\mu_\gamma}\left(v(E(\varphi))B^n\right) f d\mu_\gamma\\ &= \int_{H^{1-\alpha,s}} \left[v(E(\varphi))\delta_{\mu_\gamma}B^n - v'(E(\varphi)) < B^n, \nabla E(\varphi) >_{2,s}\right] f d\mu_\gamma\\ &= 0,\end{aligned}$$

because, as we saw in Theorem 2.3.1, $\delta_{\mu_\gamma} B^n = 0$ and

$$< B^n, \nabla E(\varphi) >_{2,s} = 2 < B(\varphi^n), \varphi^n >_{1,s} = 0$$

since the energy is conserved. $\square$

In order to prove existence of a global averaged-Euler flow defined ν_γ^r almost everywhere and taking values on the level sets of the energy E, recall the finite dimensional result of Lemma 2.3.1 and that we denoted the flow associated to B^n on $H^{1-\alpha,s}$ by $U^n(t,\varphi) = U^n(t,\varphi^n) + \Pi_n^\perp \varphi$.

Theorem 2.5.2. *Let $\alpha > \frac{2-s}{1+s}$. For all $r > 0$ such that $\rho(r) > 0$, there exists a flow $U'(\cdot,\omega)$ defined on a probability space $(\Omega, \mathcal{F}, P_\gamma^r)$ with values in V_r, $U'(\cdot,\omega) \in C(\mathbb{R}; V_r)$, $\omega \in \Omega$ such that:*

1. *for any B^* redefinition of B,*

$$U_k'(t,\omega) = U_k'(0,\omega) + \int_0^t B_k^*(U'(s,\omega))ds, \quad P_\gamma^r - a.e.\,\omega,\ \forall t \in \mathbb{R},$$

2. *ν_γ^r is invariant for the flow, in the sense that:*

$$\int f(U'(t,\omega))dP_\gamma^r(\omega) = \int f(\varphi)d\nu_\gamma^r(\varphi), \qquad \forall t \in \mathbb{R},\ \forall f \in \mathcal{D}.$$

Before the proof of the theorem we give a complementary Lemma.

Lemma 2.5.1. *The approximated averaged-Euler vector field B^n converges to B in $L^2_{\nu_\gamma^r}(H^{1-\alpha,s}; H^{1-\alpha,s})$ as $n \to \infty$.*

Proof. From the results on the regularity of B, Subsection 2.3.1, we get $B \in W_\infty(H^{1-\alpha,s})$ ($W_\infty(H^{1-\alpha,s})$ denotes the space of functions W_∞ with values in $H^{1-\alpha,s}$) and therefore

$$\int_{V_r} (\|B(\varphi)\|^2_{1-\alpha,s})^* d\nu_\gamma^r < \infty. \tag{2.19}$$

Also from the results of Subsection 2.3.1 it follows that B^n is a Cauchy sequence in $W_r^p(H^{1-\alpha,s})$ for all r, p and by definition of ν_γ^r we have

$$\rho(r)\int_{V_r} (\|B^n - B\|^2_{1-\alpha,s})^* d\nu_\gamma^r = \rho_{\|B^n-B\|^2_{1-\alpha,s}}(r).$$

Hence if we show that $\rho_{\|B^n-B\|^2_{1-\alpha,s}}(r)$ converges to zero as n tends to infinity, we get the lemma. From [1, 53] we know that $\rho_{\|B^n-B\|^2_{1-\alpha,s}}(r) \le \left\| \|B^n - B\|^2_{1-\alpha,s} \right\|_{W_r^p}$ while

$$\left\| \|B^n - B\|^2_{1-\alpha,s} \right\|_{W_r^p} \le C\|B^n - B\|^2_{W_r^{2p}(H^{1-\alpha,s})}.$$

In fact we have

$$|D_{\hat{e}_j}\|B^n - B\|^2_{1-\alpha,s}| \leq 2\|D_{\hat{e}_j}(B^n - B)\|_{1-\alpha,s}\|B^n - B\|_{1-\alpha,s}$$

and thus

$$\|\nabla\|B^n - B\|^2_{1-\alpha,s}\|_{2,s} \leq 2\|B^n - B\|_{1-\alpha,s}\|\nabla(B^n - B)\|_{H.S.(H^{2,s};H^{1-\alpha,s})}.$$

A similar argument holds for the higher order derivatives. □

In particular, from Lemma 2.5.1, there exists a constant C_2 such that

$$\sup_n \int_{V_r} (\|B^n(\varphi)\|^2_{1-\alpha,s})^* d\nu^r_\gamma \leq C_2, \quad \forall\alpha > \frac{2-s}{1+s}.$$

We finally prove Theorem 2.5.2

Proof of Theorem 2.5.2. Let $t \in \mathbb{R}^+$ and consider U^n_k as a stochastic process with laws on the space $C(\mathbb{R}^+;\mathbb{C})$ endowed with the sup-norm:

$$\eta^n_k(\Gamma) = \nu^r_\gamma(\{\varphi : U^n_k(\cdot,\varphi) \in \Gamma\}), \quad \Gamma \subset C(\mathbb{R}^+;\mathbb{C}).$$

We consider the weak topology on the space of measures on $C(\mathbb{R}^+;\mathbb{C})$. We have

1. $$\eta^n_k(|y(0)| > R) \leq \frac{1}{R^2}\mathbb{E}_{\nu^r_\gamma}|\omega_k|^2 \leq \frac{C_3}{R^2} \to 0 \text{ when } R \to \infty$$

2. for all $L > 0$ and $T > 0$,

$$\begin{aligned}\eta^n_k\left(\sup_{\substack{0\leq t\leq t'\leq T\\ t'-t>\delta}} |y(t') - y(t)| > L\right) &\leq \frac{1}{L^2}\mathbb{E}_{\nu^r_\gamma}\left(\sup_{t',t}|U^n_k(t',\varphi) - U^n_k(t,\varphi)|^2\right)\\ &\leq \frac{\delta}{L^2}\mathbb{E}_{\nu^r_\gamma}\left(\int_0^T |B^n_k(U^n(s,\varphi))|^2 ds\right)\\ &\leq \frac{T\delta}{L^2}\mathbb{E}_{\nu^r_\gamma}|B^n_k|^2\\ &\leq \frac{T\delta C_2}{L^2} \to 0 \text{ when } \delta \to 0,\end{aligned}$$

where in the last inequalities we used respectively that B^n is ν^r_γ-invariant for all n and that $\sup_n \mathbb{E}_{\nu^r_\gamma}\|B^n(\varphi)\|^2_{1-\alpha,s} \leq C_2$. Hence, by Prohorov's criterium (actually a combined version of Prohorov's criterium and Ascoli-Arzelà theorem) we can state that there exists a subsequence of η^n_k (for simplicity it will also be denoted by η^n_k), that converges to some probability measure η_k. We denote by U_k the stochastic process with law η_k. By Skorohod's theorem, there exists a probability space $(\Omega, \mathcal{F}, P_\gamma^r)$ and a family of processes $U'^n(t,\omega)$, $U'(t,\omega)$ with laws respectively η^n, η. Furthermore $U'^n(\cdot,\omega) \to U'(\cdot,\omega)$ for a.e. $\omega \in \Omega$, that is, there exists $A \subset \Omega$ such that $P_\gamma^r(A^c) = 0$ and for all $\omega \in A$, $U'^n(s,\omega) \to U'(s,\omega)$ in $H^{1-\alpha,s}$ for all $s \in [0,T]$. Repeating the construction for the processes $t \in \mathbb{R}^+ \mapsto U_k^n(-t,\varphi)$ we obtain the negative values of t. We now prove 2: for all f in $\mathcal{D}$, because $\delta_{\nu_\gamma^r} B^n = 0$ for all n (Theorem 2.5.1), we have,

$$\int f(U^n(t,\varphi))d\nu_\gamma^r(\varphi) = \int f(\varphi)d\nu_\gamma^r(\varphi).$$

On the other hand, by definition of η^n and η,

$$\begin{aligned}\int f(U^n(t,\varphi))d\nu_\gamma^r(\varphi) &= \int f(y(t))d\eta^n(y(t)) \\ &= \int f(U'^n(t,\omega))dP_\gamma^r(\omega) \\ &\to \int f(U'(t,\omega))dP_\gamma^r(\omega) \text{ when } n \to \infty.\end{aligned}$$

To prove 1 it is enough to check that

$$\int \left| \int_0^t B_k^n(U'^n(s,\omega)) - B_k^*(U'(s,\omega))ds \right| dP_\gamma^r$$

converges to zero when n goes to infinity; we have

$$\begin{aligned}\int \left| \int_0^t B_k^n(U'^n(s,\omega)) - B_k^*(U'(s,\omega))ds \right| dP_\gamma^r &\le \int_0^T \int \left| B_k^n(U'^n(s,\omega)) - B_k^*(U'^n(s,\omega)) \right| dP_\gamma^r ds \\ &+ \int_0^T \int \left| B_k^*(U'^n(s,\omega)) - B_k^*(U'(s,\omega)) \right| dP_\gamma^r ds.\end{aligned}$$

The first integral converges to zero by the ν_γ^r-invariance of the flow and because B^n converges to B in $L^2_{\nu_\gamma^r}(H^{1-\alpha,s}; H^{1-\alpha,s})$ as we proved in Lemma 2.5.1. For the second integral consider $D' = [0,T] \times A^c$, clearly $\lambda \times P_\gamma^r(D') = 0$ where $\lambda(ds)$ denotes the Lebesgue measure on $\mathbb{R}$. Also we can find a subset $D \subset H^{1-\alpha,s}$ such that $\nu_\gamma^r(D) = 0$ and B^* restricted to D^c is continuous, then define

$$A_n = \{(s,\omega) \in [0,T] \times \Omega : U'^n(s,\omega) \in D\}$$

and

$$A_\infty = \{(s,\omega) \in [0,T] \times \Omega : U'(s,\omega) \in D\}.$$

We have $\lambda \times P_\gamma^r(A_n) = 0$ for all n and $\lambda \times P_\gamma^r(A_\infty) = 0$. Set

$$\Delta = A_\infty \cup (\cup_n A_n) \cup D'.$$

Let $(s,\omega)\in\Delta^c$; in particular U'^n and U' take values in D^c (in which B^* is continuos) and $U'^n_k(s,\omega)\to U'_k(s,\omega)$ in $\mathbb{C}$. We have,

$$\left|B^*_k(U'^n(s,\omega))-B^*_k(U'(s,\omega))\right|\to 0$$

and since

$$\int_0^T\int\left|B^*_k(U'^n(s,\omega))-B^*_k(U'(s,\omega))\right|dP^r_\gamma ds$$

is uniformly bounded, by Egoroff's theorem the second integral also converges to zero. □

2.5.2 Return to a neighborhood of its initial state

The Poincaré recurrence theorem holds in our particular case. This is used here to prove that the globally defined invariant flow returns infinitely many times in a neighborhood of the initial state. A similar result is proved in [28] for the one-dimensional Camassa-Holm equation and certain initial profiles for which the solutions exist globally. We recall the Poincaré recurrence theorem (c.f. [47]).

Theorem 2.5.3. *Let P be a probability measure defined on a set Ω. If $\{T_t\}_{t\geq 0}$ is a one-parameter family of measure preserving transformations and Ω_0 is a subset of Ω with $P(\Omega_0)>0$, then for P-almost every $\omega\in\Omega_0$ there exist arbitrarily large t such that $T_t\omega\in\Omega_0$.*

Theorem 2.5.4. *Let $\alpha>\frac{2-s}{1+s}$ and fix $\varphi_0\in V_r\subset H^{1-\alpha,s}$. If $\varepsilon>0$ is sufficiently small, then for ν^r_γ-a.e. $\varphi\in V_r\subset H^{1-\alpha,s}$ such that $\|\varphi-\varphi_0\|_{1-\alpha,s}<\varepsilon$, there exists a sequence $\{t_n\}\uparrow\infty$ such that the corresponding invariant flow starting from φ, $U_\varphi(t,\omega)$, satisfies $\mathbb{E}_{P^r_\gamma}\|U_\varphi(t_n,\omega)-\varphi_0\|_{1-\alpha,s}<2\varepsilon$.*

Proof. The statement follows by applying Poincaré recurrence theorem to the open set $B(\varphi_0,\varepsilon)=\{\varphi\in V_r\subset H^{1-\alpha,s}:\|\varphi-\varphi_0\|^2_{1-\alpha,s}<\varepsilon\}$ with $\nu^r_\gamma\{B(\varphi_0,\varepsilon)\}>0$. □

Chapter 3

Invariant measures for the non-periodic two-dimensional Euler equations

3.1 Introduction

Euler equation describes the time evolution of an incompressible non-viscous fluid with constant density. This fundamental equation has been and still is intensively studied. Among the numerous references on the Euler equation, we cite the books [13, 51, 55]. It is known that solutions do not blow up starting from smooth data with finite kinetic energy (T. Kato (1967) [48], C. Bardos (1972) [14] among others). In two dimensions, for bounded domains and when the initial vorticity is bounded, existence, uniqueness and global regularity of solutions was shown (V.I. Judovic, 1963 [46]); these results were extended, in the framework of weak solutions, to the case where the initial vorticity belongs to L^p, with $p > 1$ and even for $p = 1$, when the vorticity is some finite measure.

A more geometric approach, identifying the solutions of the Euler equation with velocities of geodesics in a space of diffeomorphisms of the underlying state space, was initiated by V. Arnold (1966) [12]. It allowed to show existence of local solutions in some Sobolev spaces (D. G. Ebin and J. Marsden, 1970 [37]).

Much less is known about irregular solutions of the Euler equation. This paper is devoted to a class of such solutions.

In statistical approaches to hydrodynamics, discussed in the physics literature on turbulence, one considers the evolution of probability densities instead of pointwise solutions. A major subject of interest is the search for invariant measures. In particular such measures are important because they can be used to prove the existence and study properties of Euler flows defined almost-everywhere with respect to them.

In this paper we extend the work [3] in two dimensions to the non-periodic setting.

We prove the existence of invariant probability measures for the Euler flow and show the existence of these flows, for all times and in some weak sense, living in the support of the invariant measures. Those are spaces of very low regularity, namely Sobolev spaces of negative order.

In Section 3.2 we recall the characterisation of Euler equations in the periodic setting and we fix the notation. For each parameter $\gamma > 0$, we denote by $\mu_{L,\gamma}$ the invariant measure for the two-dimensional Euler flow on $[0, L]^2$. These measures $\mu_{L,\gamma}$ were previously constructed in [3]. In Section 3.3, we show the weak convergence of $\mu_{L,\gamma}$ to some μ_γ in $H^\beta_{loc}(\mathbb{R}^2)$ for $\beta < 1$ when the period L tends to infinity. We follow similar arguments to those used in [35] for the Klein-Gordon equation in dimension one. Here we also show that $H^\beta_{loc}(\mathbb{R}^2)$ for $\beta < 1$ is the support of μ_γ. Finally, in Section 3.4 we study the $L^p_{\mu_\gamma}$-regularity of the vector field, B, and prove the existence of a globally defined Euler flow, U, under which the probability measures μ_γ are invariant. We conclude this section by proving the continuity of the flow.

3.2 The 2D Euler equations

Consider the incompressible non-viscous Euler equations on $\mathbb{R}^2$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \qquad \nabla \cdot u = 0 \tag{3.1}$$

where $u : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ denotes the time dependent velocity field and $p : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ denotes the pressure. The first equation is Newton's second law (the acceleration is proportional to the pressure) and the second equation is the incompressibility condition.

We have the following,

Theorem 3.2.1. *The time dependent vector field u is a smooth solution of (3.1) if and only if there exists a smooth (real) function φ (stream function) such that $u = \nabla^\perp\varphi$ and φ is a solution of the equation*

$$\frac{\partial \Delta\varphi}{\partial t} = -(\nabla^\perp\varphi \cdot \nabla)\Delta\varphi. \tag{3.2}$$

Proof. We refer to [4]. □

Here $\nabla^\perp\varphi = (-\partial_2\varphi, \partial_1\varphi)$, where ∂_1, ∂_2 denote respectively the partial derivative with respect to the first and second variable. The two problems, (3.1) and (3.2), are equivalent; below we consider (3.2).

3.2.1 Periodic case

We recall here the most relevant results from [3] about the periodic case. On the space $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2 \simeq [0, L]^2$ such that $L > 0$ denotes the period, consider equation (3.2) with periodic boundary condition

$$\varphi(0, y, t) = \varphi(L, y, t) \text{ and } \varphi(x, 0, t) = \varphi(x, L, t), \qquad \forall\ (x, y) \in \mathbb{T}^2.$$

In [3] is considered the case $L = 2\pi$, but the analysis for general $L > 0$ is identical if we simply re-scale.

The energy and the enstrophy, namely $E(u) = \frac{1}{2}\int_{\mathbb{T}^2} |u|^2 dx$ and $S(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\text{curl } u|^2 dx$, are conserved by the Euler velocity. In terms of the stream function φ they can be written as

$$E(\varphi) = -\frac{1}{2}\int_{\mathbb{T}^2} \varphi \Delta \varphi dx$$

and

$$S(\varphi) = \frac{1}{2}\int_{\mathbb{T}^2} |\Delta \varphi|^2 dx.$$

We denote by $\{e_k^L\}_{k \in \mathbb{Z}^2}$ the following orthonormal basis of $L^2(\mathbb{T}^2)$,

$$e_k^L = \frac{1}{L} e^{i\frac{2\pi}{L} k \cdot x}, \quad \forall\ k \in \mathbb{Z}^2.$$

For all real-valued $u \in L^2(\mathbb{T}^2)$ we have

$$u(x, t) = \sum_{k>0} u_k^L(t) e_k^L(x),$$

where we say that $k = (k_1, k_2) \in \mathbb{Z}^2$ is positive if $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$. This may be assumed since, on one hand we deal with real-valued functions ($u_{-k} = \bar{u}_k$) and, on the other hand, we can consider, without loss of generality, these functions to have zero mean. We identify the Sobolev spaces $H^\beta(\mathbb{T}^2)$ of real-valued functions with

$$H^\beta := \left\{ u = \sum_{k>0} u_k^L e_k^L : \sum_{k>0} \left(\frac{2\pi k}{L}\right)^{2\beta} |u_k^L|^2 < +\infty \right\}, \tag{3.3}$$

where by k^2 we denote the inner product $k \cdot k = k_1^2 + k_2^2$ and $|k| = \sqrt{k^2}$. Through the paper the powers of k are denoted by $|k|^\beta$ when β is odd, and, with a slight abuse of notation, k^β when β is even.

For all β, H^β is a Hilbert space with inner product given by

$$< u, v >_\beta := \sum_{k>0} \left(\frac{2\pi k}{L}\right)^{2\beta} u_k^L \bar{v}_k^L.$$

Using the expansion on the $L^2([0,L]^2)$ basis, for $\varphi^L(x,t) = \sum_{k>0} \varphi_k^L(t) e_k^L(x)$ the equations reduce to an infinite dimensional system of first order ODEs

$$\frac{d}{dt}\varphi_k^L(t) = B_k^L(\varphi^L), \qquad \forall\ k \in \mathbb{Z}^2 \tag{3.4}$$

where

$$B_L(\varphi^L) := \sum_{k>0} B_k^L(\varphi^L) e_k^L(x) \tag{3.5}$$

and

$$B_k^L(\varphi^L) = \frac{1}{L}\left(\frac{2\pi}{L}\right)^2 \sum_{\substack{h>0\\ h\neq k}} \left[\frac{(h^\perp\cdot k)(k\cdot h)}{k^2} - \frac{h^\perp\cdot k}{2}\right] \varphi_h^L \varphi_{k-h}^L, \tag{3.6}$$

where $h^\perp = (-h_2, h_1)$. We write $B_k^L(\varphi^L) = \sum_h \alpha_{h,k}^L \varphi_h^L \varphi_{k-h}^L$, with

$$\alpha_{h,k}^L = \frac{1}{L}\left(\frac{2\pi}{L}\right)^2 \left[\frac{(h^\perp\cdot k)(k\cdot h)}{k^2} - \frac{h^\perp\cdot k}{2}\right]. \tag{3.7}$$

3.2.2 Notations

Let us consider some relevant function spaces that will be used below. For all $\beta \in \mathbb{R}$ we define the local Sobolev spaces $H^\beta_{loc}(\mathbb{R}^2)$ by

$$H^\beta_{loc}(\mathbb{R}^2) := \{\text{real-valued } u : \forall K \subset \mathbb{R}^2 \text{ compact},\ D^\beta u \in L^2(K)\},$$

where the operator D^β is considered as a pseudo-differential operator. We may assume that the compact sets K are of the type $K = [0,L]\times[0,L]$ for $L \in \mathbb{N}^*$. The spaces $H^\beta_{loc}(\mathbb{R}^2)$ are not normed spaces, however it is possible to equip them with the topology induced by the distances $d_{\beta,2}$ defined by

$$d_{\beta,2}(u,v) := \sum_{L\in\mathbb{N}^*} 2^{-L} \frac{\|D^\beta(u-v)\|_{L^2([0,L]^2)}}{1+\|D^\beta(u-v)\|_{L^2([0,L]^2)}}. \tag{3.8}$$

In particular the metric spaces $\left(H^\beta_{loc}(\mathbb{R}^2); d_{\beta,2}\right)$ are complete for all $\beta \in \mathbb{R}$. Analogously, for all $\beta \in \mathbb{R}$ we define the spaces $W^{\beta,\infty}_{loc}(\mathbb{R}^2)$ by

$$W^{\beta,\infty}_{loc}(\mathbb{R}^2) := \{\text{real-valued } u : \forall K \subset \mathbb{R}^2 \text{ compact},\ D^\beta u \in L^\infty(K)\}.$$

The metric spaces $W^{\beta,\infty}_{loc}(\mathbb{R}^2)$ are complete if endowed with the distances $d_{\beta,\infty}$ defined by

$$d_{\beta,\infty}(u,v) := \sum_{L\in\mathbb{N}^*} 2^{-L} \frac{\|D^\beta(u-v)\|_{L^\infty([0,L]^2)}}{1+\|D^\beta(u-v)\|_{L^\infty([0,L]^2)}}. \tag{3.9}$$

For each fixed β we have

$$W^{\beta,\infty}_{loc}(\mathbb{R}^2) \subseteq H^{\beta}_{loc}(\mathbb{R}^2).$$

We remark that, usually, the spaces $H^{\beta}_{loc}(\mathbb{R}^2)$ and $W^{\beta,\infty}_{loc}(\mathbb{R}^2)$ are characterized by functions u such that $(I-\Delta)^{\beta/2}u$ belong to $L^2(K)$ and $L^\infty(K)$, respectively, and for every compact subset K of $\mathbb{R}^2$. For further results concerning local Sobolev spaces we refer to [11].

We say that a real-valued function u belongs to the weighted Sobolev space $W^{\beta,\infty}(\mathbb{R}^2, 1+|x|)$ for some fixed $\beta \in \mathbb{R}$ if

$$\|(1+|x|)^{-1} D^\beta u\|_{L^\infty(\mathbb{R}^2)} < +\infty.$$

The following inclusion holds:

$$W^{\beta,\infty}(\mathbb{R}^2, 1+|x|) \subseteq W^{\beta,\infty}_{loc}(\mathbb{R}^2). \tag{3.10}$$

Below, we use $X \lesssim Y$ to denote the estimate $X \le CY$ for some constant C. Unless stated otherwise C is an unessential constant, in particular independent from the period L.

3.3 Invariant measures

In the periodic setting and for each parameter $\gamma \in \mathbb{R}^+$, invariant probability measures, $\mu_{L,\gamma}$, were constructed, see [3]. In this section we define measures μ_γ as the weak limits of $\mu_{L,\gamma}$ when L tends to infinity. Moreover we show that the support of μ_γ is the Sobolev space $H^\beta_{loc}(\mathbb{R}^2)$ for $\beta < 1$.

3.3.1 Approximations of μ_γ

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for each $L > 0$ and $R = (R_1, R_2) \in \mathbb{N}^2$, we consider the stochastic process

$$\Phi_{L,R}(\omega, x) := \sum_{\substack{k>0 \\ k_1<LR_1 \\ k_2<LR_2}} a^L_k(\omega) e^L_k(x),$$

where

$$a^L_k(\omega) := \chi_k(\omega)\sqrt{\frac{2}{\gamma}}\left(\frac{L}{2\pi k}\right)^2,$$

and $\{\chi_k\}_{k\in\mathbb{Z}^2}$ denotes a sequence of complex-valued i.i.d. Gaussian random variables. Therefore, for all fixed k, $a^L_k(\omega)$ denotes a complex-valued random variable with mean zero and variance $\frac{2}{\gamma}\left(\frac{L}{2\pi k}\right)^4$ and $\Phi_{L,R}$ is a Gaussian vector with law and covariance matrix given respectively by,

$$(\det M(L))^{-1/2} e^{-<a, M(L)^{-1}a>} \prod_{\substack{k>0 \\ k_1<LR_1 \\ k_2<LR_2}} \gamma \frac{da^L_k(\omega)}{2\pi}$$

and

$$M(L)_{k,j} = \mathbb{E}_{\mathbb{P}}(a_k^L \bar{a}_j^L) = \delta_j^k \frac{2}{\gamma} \left(\frac{L}{2\pi k}\right)^4,$$

where δ_j^k is the Kronecker symbol, so that

$$< a, M(L)^{-1} a >= \sum_{\substack{k>0\\ k_1<LR_1\\ k_2<LR_2}} \left(\frac{2}{\gamma}\left(\frac{L}{2\pi k}\right)^4\right)^{-1} |a_k^L(\omega)|^2.$$

Remark that, if

$$\varphi^{L,R}(x) = \sum_{\substack{k>0\\ k_1<LR_1\\ k_2<LR_2}} \varphi_k^{L,R} e_k^L(x),$$

then

$$\sum_{\substack{k>0\\ k_1<LR_1\\ k_2<LR_2}} \left(\frac{2}{\gamma}\left(\frac{L}{2\pi k}\right)^4\right)^{-1} |\varphi_k^{L,R}|^2 = \frac{\gamma}{2}\int_{\mathbb{T}^2} |\Delta\varphi^{L,R}|^2 dx;$$

that is

$$< \varphi^{L,R}, M(L)^{-1}\varphi^{L,R} >= S(\varphi^{L,R}),$$

where by $S(\varphi^{L,R})$ we denoted the enstrophy. Hence the measure $d\mu_{L,\gamma}$, formally defined by

$$d\mu_{L,\gamma}(\varphi^L) := e^{-\frac{\gamma}{2}\int_{\mathbb{T}^2}|\Delta\varphi^L|^2 dx}\mathcal{D}\varphi^L, \quad \mathcal{D}\varphi^L = \prod_{k>0}\gamma\left(\frac{2\pi k}{L}\right)^4 \frac{d\varphi_k^L}{2\pi} \tag{3.11}$$

is the law of Φ_L on some Banach space, where

$$\Phi_L(\omega, x) := \sum_{k>0} a_k^L(\omega) e_k^L(x). \tag{3.12}$$

The measure $\mu_{L,\gamma}$ coincides with the Gibbs-type measure, relative to the enstrophy, defined in [3]. It was proved in [3] that $(H^\beta, H^2, \mu_{L,\gamma})$ is a complex abstract Wiener space for $\beta < 1$; in particular H^2 is a densely embedded Hilbert subspace of the Banach space H^β and $\mu_{L,\gamma}$ is a Gaussian measure with

$$\int e^{i\gamma l(\varphi^L)} d\mu_{L,\gamma}(\varphi^L) = e^{-\frac{1}{2}\gamma\|l\|_2^2}, \qquad \forall\, l \in (H^\beta)' \subset H^2.$$

The space H^β denotes the support of $\mu_{L,\gamma}$ and H^2 the associated Cameron-Martin space.

Remark 3.3.1. By definition, the support of $\mu_{L,\gamma}$ is the space in which the random variable Φ_L takes values $\mathbb{P}-$a.e.. Hence, in particular, Φ_L takes values on real valued functions, as it should, since we are dealing with Euler equations. The same is true in the limit of L that tends to infinity, see Proposition 3.3.2 below.

Below, we define Φ as the limit in $L^2(\Omega; H^\beta_{loc}(\mathbb{R}^2))$ of the sequence of random variables $\{\Phi_L\}_{L\in\mathbb{N}^*}$ given in equation (3.12) and we define the measure μ_γ on functions of $\mathbb{R}^2$ as the image measure under the random variable Φ. We follow the ideas of [35] where the Klein-Gordon equation on the real line is considered.

Proposition 3.3.1. *The sequence $\{\Phi_L\}_{L\in\mathbb{N}^*}$ is a Cauchy sequence in $L^2(\Omega; H^\beta_{loc}(\mathbb{R}^2))$ for $\beta < 1$.*

Proof. First observe that

$$W^{\beta,\infty}(\mathbb{R}^2) \subseteq W^{\beta,\infty}(\mathbb{R}^2, 1+|x|) \subseteq W^{\beta,\infty}_{loc}(\mathbb{R}^2) \subseteq H^\beta_{loc}(\mathbb{R}^2)$$

and that we can write for $0 < L < S$

$$\Phi_L - \Phi_S = \Phi_L - \Phi_{L,R} + \Phi_{L,R} - \Phi_{S,R} + \Phi_{S,R} - \Phi_S.$$

We will show that $\mathbb{E}_\mathbb{P}\|D^\beta(\Phi_L - \Phi_{L,R})\|^2_{L^\infty(\mathbb{R}^2)}$ converges to zero when R tends to infinity uniformly in L and that $\mathbb{E}_\mathbb{P}\|(1+|x|)^{-1}D^\beta(\Phi_{L,R} - \Phi_{S,R})\|^2_{L^\infty(\mathbb{R}^2)}$ tends to zero when L tends to infinity uniformly in R. We have

$$\begin{aligned}
\mathbb{E}_\mathbb{P}\|D^\beta(\Phi_L - \Phi_{L,R})\|^2_{L^\infty(\mathbb{R}^2)} &= \mathbb{E}_\mathbb{P}\left[\sup_{x\in\mathbb{R}^2}\left|\sum_{\substack{k_1\geq LR_1\\ k_2\geq LR_2}}\left(\frac{2\pi|k|}{L}\right)^{\beta-2}\chi_k(\omega)\sqrt{\frac{2}{\gamma}}e^L_k(x)\right|\right]^2 \\
&\leq \mathbb{E}_\mathbb{P}\left[\frac{1}{L}\sum_{\substack{k_1\geq LR_1\\ k_2\geq LR_2}}\left(\frac{2\pi|k|}{L}\right)^{\beta-2}|\chi_k(\omega)|\sqrt{\frac{2}{\gamma}}\right]^2 \\
&= \frac{1}{L^2}\frac{2}{\gamma}\sum_{\substack{k_1\geq LR_1\\ k_2\geq LR_2}}\sum_{\substack{h_1\geq LR_1\\ h_2\geq LR_2}}\left(\frac{2\pi|k|}{L}\right)^{\beta-2}\left(\frac{2\pi|h|}{L}\right)^{\beta-2}\mathbb{E}_\mathbb{P}[\chi_k(\omega)\bar{\chi}_h(\omega)] \\
&\leq \frac{2}{\gamma}\sum_{\substack{k_1\geq LR_1\\ k_2\geq LR_2}}\left(\frac{2\pi k}{L}\right)^{2\beta-4} \\
&\lesssim \int_{[R,+\infty)^2}\frac{dy}{y^{4-2\beta}} \leq \varepsilon
\end{aligned}$$

for R sufficiently big and uniformly in L, since $\beta < 1$.

Now suppose that $L = 2^n$ and $S = 2^m$ with $n < m$; we have

$$D^\beta(\Phi_{2^n,R} - \Phi_{2^m,R}) = \sqrt{\frac{2}{\gamma}} \Bigg[\sum_{\substack{k>0 \\ k_1<2^n R_1 \\ k_2<2^n R_2}} \left(\frac{2\pi|k|}{2^n}\right)^{\beta-2} \chi_k(\omega) 2^{-n} e^{i\frac{2\pi}{2^n}k\cdot x} \tag{3.13}$$

$$- \sum_{\substack{l>0 \\ l_1<2^m R_1 \\ l_2<2^m R_2}} \left(\frac{2\pi|l|}{2^m}\right)^{\beta-2} \chi_l(\omega) 2^{-m} e^{i\frac{2\pi}{2^m}l\cdot x} \Bigg]. \tag{3.14}$$

Also, for a sequence of complex-valued i.i.d. Brownian motions, $\{W_{k^2}\}_{k\in\mathbb{Z}^2}$,

$$\chi_l(\omega)2^{-m} \simeq W_{\frac{l^2+1}{2^{2m}}}(\omega) - W_{\frac{l^2}{2^{2m}}}(\omega),$$

where here $\simeq$ denotes identification in law. In particular, $\chi_l(\omega)2^{-m}$ can be written as

$$\chi_l(\omega)2^{-m} \simeq \sum_{j=0}^{2^{n-m}-1} \chi_{2^{n-m}l+j}(\omega)2^{-n}, \tag{3.15}$$

where we define the sum $2^{n-m}l + j := (2^{n-m}l_1 + j; 2^{n-m}l_2 + j)$ for any $l = (l_1, l_2) \in \mathbb{Z}^2$ and $j \in \{0, \cdots, 2^{n-m} - 1\}$, indeed

$$\begin{aligned} \sum_{j=0}^{2^{n-m}-1} \chi_{2^{n-m}l+j}(\omega)2^{-n} &= \sum_{j=0}^{2^{n-m}-1} W_{\frac{(2^{n-m}l+j)^2+1}{2^{2n}}}(\omega) - W_{\frac{(2^{n-m}l+j)^2}{2^{2n}}}(\omega) \\ &\simeq W_{\frac{l^2+1}{2^{2m}}}(\omega) - W_{\frac{l^2}{2^{2m}}}(\omega) \\ &= \chi_l(\omega)2^{-m}. \end{aligned}$$

Therefore

$$D^\beta(\Phi_{2^n,R} - \Phi_{2^m,R}) \simeq$$

$$\simeq \sqrt{\frac{2}{\gamma}} \sum_{\substack{l>0 \\ l_1<2^m R_1 \\ l_2<2^m R_2}} \sum_{j=0}^{2^{n-m}-1} \chi_{2^{n-m}l+j}(\omega)2^{-n} \left[\frac{e^{i2\pi\frac{(2^{n-m}l+j)}{2^n}\cdot x}}{\left(\frac{|2^{n-m}l+j|}{2^n}\right)^{2-\beta}} - \frac{e^{i2\pi\frac{l}{2^m}\cdot x}}{\left(\frac{|l|}{2^m}\right)^{2-\beta}} \right].$$

To get the last equality (in law) we used: in (3.13) the change of variable $k = 2^{n-m}l + j$; and in (3.14) the replacement of (3.15).

Take the $L^2(\Omega)$ norm of $D^\beta(\Phi_{2^n,R} - \Phi_{2^m,R})$:

$$\mathbb{E}_\mathbb{P}|D^\beta(\Phi_{2^n,R} - \Phi_{2^m,R})|^2 \lesssim \sum_{\substack{l>0 \\ l_1<2^m R_1 \\ l_2<2^m R_2}} \sum_{j=0}^{2^{n-m}-1} 2^{-2n} \left[\frac{e^{i2\pi\frac{(2^{n-m}l+j)}{2^n}\cdot x}}{\left(\frac{|2^{n-m}l+j|}{2^n}\right)^{2-\beta}} - \frac{e^{i2\pi\frac{l}{2^m}\cdot x}}{\left(\frac{|l|}{2^m}\right)^{2-\beta}} \right]^2$$

and use that the directional derivatives of the function $y \in \mathbb{R}^2 \mapsto \frac{e^{i2\pi y\cdot x}}{|y|^{2-\beta}}$ are bounded by $C(\beta)\frac{(1+2\pi|x|)}{|y|^{2-\beta}}$ in order to obtain

$$\sum_{\substack{l>0\\ l_1<2^m R_1\\ l_2<2^m R_2}} \sum_{j=0}^{2^{n-m}-1} 2^{-2n} \left[\frac{e^{i2\pi \frac{(2^{n-m}l+j)}{2^n}\cdot x}}{\left(\frac{|2^{n-m}l+j|}{2^n}\right)^{2-\beta}} - \frac{e^{i2\pi \frac{l}{2^m}\cdot x}}{\left(\frac{|l|}{2^m}\right)^{2-\beta}} \right]^2$$
$$\lesssim \sum_{\substack{l>0\\ l_1<2^m R_1\\ l_2<2^m R_2}} \sum_{j=0}^{2^{n-m}-1} 2^{-2n} \frac{(1+2\pi|x|)^2}{\left(\frac{l}{2^m}\right)^{4-2\beta}} \left(\frac{j}{2^n}\right)^2 .$$

Use the inequality

$$\sum_{j=0}^{2^{n-m}-1} \left(\frac{j}{2^n}\right)^2 \leq \sum_{j=0}^{2^{n-m}} 2^{-2m} = 2^{n-3m}$$

to get

$$\sum_{\substack{l>0\\ l_1<2^m R_1\\ l_2<2^m R_2}} 2^{n-3m} \frac{(1+2\pi|x|)^2}{\left(\frac{l}{2^m}\right)^{4-2\beta}} \lesssim \varepsilon(1+|x|)^2 \int_{[a,+\infty)^2} \frac{dy}{y^{4-2\beta}} \lesssim \varepsilon(1+|x|)^2$$

for m sufficiently big and uniformly in R since $\beta < 1$ and $a \in (0,1)$. Back to L and S we have $\mathbb{E}_{\mathbb{P}}\|(1+|x|)^{-1}D^\beta(\Phi_{L,R} - \Phi_{S,R})\|^2_{L^\infty(\mathbb{R}^2)} \leq \varepsilon$ for L sufficiently big and uniformly in R. $\square$

In the following we denote by μ_γ the law of Φ where Φ is the limit of $\{\Phi_L\}_{L\in\mathbb{N}^*}$ in $L^2(\Omega; H^\beta_{loc}(\mathbb{R}^2))$. This L^2-convergence implies that $\mu_{L,\gamma}$ converges weakly to μ_γ in $H^\beta_{loc}(\mathbb{R}^2)$ when L tends to infinity.

3.3.2 Support of μ_γ

We study the support of the measure μ_γ. Since μ_γ is the law of Φ, its support is defined as the space in which $\Phi(\omega,\cdot)$ takes values $\mathbb{P}$-almost surely.

Proposition 3.3.2. *Let $\beta < 1$, we have*

$$\mathrm{supp}(\mu_\gamma) = H^\beta_{loc}(\mathbb{R}^2).$$

Proof. We have

$$\mathbb{E}_{\mathbb{P}} d_{\beta,2}(\Phi, 0) \leq \mathbb{E}_{\mathbb{P}} d_{\beta,2}(\Phi, \Phi_{L,R}) + \mathbb{E}_{\mathbb{P}} d_{\beta,2}(\Phi_{L,R}, 0),$$

where $d_{\beta,2}$ denotes the metric for $H^{\beta}_{loc}(\mathbb{R}^2)$. On one hand and by Proposition 3.3.1, $\mathbb{E}_{\mathbb{P}} d_{\beta,2}(\Phi, \Phi_{L,R})$ tends to zero when L and R tend to infinity. On the other $\mathbb{E}_{\mathbb{P}} d_{\beta,2}(\Phi_{L,R}, 0) \leq C < +\infty$ since we have

$$\mathbb{E}_{\mathbb{P}} d_{\beta,2}(\Phi_{L,R}, 0) \leq \sum_{L} 2^{-L} \mathbb{E}_{\mathbb{P}} \|D^{\beta}\Phi_{L,R}\|_{L^2([0,L]^2)} \leq C \sum_{L} 2^{-L} < +\infty.$$

We have used the fact that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}} \|D^{\beta}\Phi_{L,R}\|^2_{L^2([0,L]^2)} &= \sum_{\substack{k>0 \\ k_1<LR_1 \\ k_2<LR_2}} \left(\frac{2\pi k}{L}\right)^{2\beta} \mathbb{E}_{\mathbb{P}} |a^L_k(\omega)|^2 \\
&\lesssim \sum_{\substack{k>0 \\ k_1<LR_1 \\ k_2<LR_2}} \left(\frac{k}{L}\right)^{2\beta-4} \\
&\lesssim \int_{[a,+\infty)^2} \frac{dy}{y^{4-2\beta}} \leq C < +\infty
\end{aligned}$$

for $a > 0$ small enough. □

Formally the measure μ_γ is given by

$$d\mu_\gamma(\varphi) = \frac{1}{Z} e^{-\frac{\gamma}{2}\int_{\mathbb{R}^2} |\Delta\varphi|^2 dx} \mathcal{D}\varphi \tag{3.16}$$

where Z is a suitable renormalizing constant. For all fixed $L \in \mathbb{N}^*$, the measure μ_γ on functions restricted to the compact phase space $[0, L]^2$ is in fact the measure $\mu_{L,\gamma}$. As in [3] for $(H^\beta, H^2, \mu_{L,\gamma})$ we can show that $(H^{\beta}_{loc}(\mathbb{R}^2), H^2_{loc}(\mathbb{R}^2), \mu_\gamma)$ is a complex abstract Wiener space for $\beta < 1$.

3.4 The velocity flow on $\mathbb{R}^2$

The aim of this section is to prove global existence and uniqueness of the Euler flow on the plane, under which μ_γ is invariant. We start with some properties of the vector field B_L in the periodic setting, given by equations (3.5)-(3.6) and previously derived in [3].

3.4.1 Approximations of the vector field B

We recall that, on an abstract Wiener space (X, H, ν) and given a field $\Psi : X \to G$, where X is a Banach space and G a Hilbert space, the gradient (in the sense of Malliavin calculus [53]) of Ψ is defined for every $u \in X$ by

$$\nabla\Psi(u)(v) = D_v\Psi(u) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\Psi(u + \varepsilon v) - \Psi(u)], \quad v \in H$$

and the limit is taken ν-a.e. in X. Also, the divergence of $\Psi \in L^2_\nu(X;G)$ is denoted by $\delta_\nu \Psi$ and defined by

$$\int \delta_\nu \Psi \cdot f d\nu = \int (\Psi, \nabla f)_G d\nu, \quad \forall f \in \mathcal{D} \tag{3.17}$$

where $\mathcal{D}$ is the space of cylindrical functionals on X, that is depending on a finite number of coordinates, and $(\cdot,\cdot)_G$ denotes the inner product of G.

Proposition 3.4.1. *The vector field B_L is divergence-free with respect to the measure $\mu_{L,\gamma}$, that is $\delta_{\mu_{L,\gamma}} B_L = 0$.*

Proof. We refer to [3] and only remark that the conservation of the enstrophy is essential to prove the statement. □

We recall the proof of the $L^p_{\mu_{L,\gamma}}$-regularity of B_L for any $p \geq 1$, as we are interested in the dependence on the period L of such estimates. For further details see [3] or [27].

Proposition 3.4.2. *Let $\beta < -1$; then the vector field $B_L \in L^p_{\mu_{L,\gamma}}(H^\beta; H^\beta)$ for all $p \geq 1$.*

Proof. It is enough to show that $\mathbb{E}_{\mu_{L,\gamma}} \|B_L(\varphi^L)\|^{2p}_{H^\beta} < +\infty$ for all p odd. We have

$$\begin{aligned}
\mathbb{E}_{\mu_{L,\gamma}} \|B_L(\varphi^L)\|^{2p}_{H^\beta} &= \mathbb{E}_{\mu_{L,\gamma}} \left[\sum_{k>0} \left(\frac{2\pi k}{L}\right)^{2\beta} |B^L_k(\varphi^L)|^2 \right]^p \\
&\leq \left[\sum_{k>0} \left(\frac{2\pi k}{L}\right)^{2\beta} \left(\mathbb{E}_{\mu_{L,\gamma}} |B^L_k(\varphi^L)|^{2p}\right)^{1/p} \right]^p
\end{aligned}$$

From $B^L_k(\varphi^L) = \sum_h \alpha^L_{h,k} \varphi^L_h \varphi^L_{k-h}$ we have

$$\begin{aligned}
\mathbb{E}_{\mu_{L,\gamma}} |B^L_k(\varphi^L)|^{2p} &= \mathbb{E}_{\mu_{L,\gamma}} \left[\sum_{h,h'} \alpha^L_{h,k} \alpha^L_{h',k} (\varphi^L_h \varphi^L_{k-h} \bar{\varphi}^L_{h'} \bar{\varphi}^L_{k-h'}) \right]^p \\
&\leq \left[\sum_{h,h'} \alpha^L_{h,k} \alpha^L_{h',k} \left(\mathbb{E}_{\mu_{L,\gamma}} (\varphi^L_h \varphi^L_{k-h} \bar{\varphi}^L_{h'} \bar{\varphi}^L_{k-h'})^p \right)^{1/p} \right]^p \\
&= \left[2 \sum_h |\alpha^L_{h,k}|^2 \left(\mathbb{E}_{\mu_{L,\gamma}} |\varphi^L_h|^{2p} \right)^{1/p} \left(\mathbb{E}_{\mu_{L,\gamma}} |\varphi^L_{k-h}|^{2p} \right)^{1/p} \right]^p \\
&\lesssim p!^2 \left[\sum_h |\alpha^L_{h,k}|^2 \frac{L^8}{h^4 (k-h)^4} \right]^p \\
&\leq p!^2 \left[L^2 \sum_h \left[\frac{(h^\perp \cdot k)(k \cdot h)}{k^2} - \frac{h^\perp \cdot k}{2} \right]^2 \frac{1}{h^4 (k-h)^4} \right]^p \\
&\leq (L^2 C)^p < +\infty, \qquad \forall\, p \geq 1.
\end{aligned}$$

Therefore, since $\beta < -1$,

$$\mathbb{E}_{\mu_{L,\gamma}} \|B_L(\varphi^L)\|_{H^\beta}^{2p} \lesssim \left(\frac{1}{L^{2\beta-2}} \sum_{k>0} \frac{1}{k^{-2\beta}} \right)^p \leq (L^{2-2\beta} C)^p < +\infty, \qquad \forall\ p \geq 1. \tag{3.18}$$

□

Remark 3.4.1. For the vector field on $[0, L]^2$ the expression $B_L(\varphi) = \sum_k B_k^L(\varphi) e_k^L(x)$ where B_k^L is defined in (3.6) is valid. Note however that the Euler vector field does not depend on L; it is the same on every finite phase space approximation and thus B_L trivially converges to B, the Euler vector field on $\mathbb{R}^2$, when L goes to infinity.

Next we show that $B : H^\beta_{loc}(\mathbb{R}^2) \to H^\beta_{loc}(\mathbb{R}^2)$ is regular with respect to $L^p_{\mu_\gamma}$ for all $p \geq 1$.

Corollary 3.4.1. *Let $\beta < -1$, then $B \in L^p_{\mu_\gamma}(H^\beta_{loc}(\mathbb{R}^2); H^\beta_{loc}(\mathbb{R}^2))$ for all $p \geq 1$.*

Proof. We show that $E_{\mu_\gamma} |d_{\beta,2}(B(\varphi), 0)|^{2p} < +\infty$ for all p odd, where $d_{\beta,2}$ denotes the metric for $H^\beta_{loc}(\mathbb{R}^2)$. We have

$$\begin{aligned} E_{\mu_\gamma} |d_{\beta,2}(B(\varphi), 0)|^{2p} &= E_{\mu_\gamma} \left| \sum_{L \in \mathbb{N}^*} 2^{-L} \frac{\|D^\beta B(\varphi)\|_{L^2([0,L]^2)}}{1 + \|D^\beta B(\varphi)\|_{L^2([0,L]^2)}} \right|^{2p} \\ &\leq \left[\sum_{L \in \mathbb{N}^*} 2^{-L} \left(E_{\mu_\gamma} \frac{\|D^\beta B(\varphi)\|^{2p}_{L^2([0,L]^2)}}{(1 + \|D^\beta B(\varphi)\|_{L^2([0,L]^2)})^{2p}} \right)^{1/2p} \right]^{2p} \\ &\leq \left[\sum_{L \in \mathbb{N}^*} 2^{-L} \left(E_{\mu_{L,\gamma}} \|D^\beta B_L(\varphi)\|^{2p}_{L^2([0,L]^2)} \right)^{1/2p} \right]^{2p}, \end{aligned}$$

where we got the last inequality from Proposition 3.4.2. Again, from estimate (3.18) and since $\beta < -1$, we conclude

$$E_{\mu_\gamma} |d_{\beta,2}(B(\varphi), 0)|^{2p} \lesssim \left[\sum_{L \in \mathbb{N}^*} 2^{-L} L^{2-2\beta} \right]^{2p} < +\infty, \qquad \forall\ p \geq 1.$$

□

In the next Lemma we prove existence for the approximated Euler equations.

Lemma 3.4.1. *For any fixed $L \in \mathbb{N}^*$ and $R \in \mathbb{N}^2$ we consider a phase space projection on $[0, L]^2$ and a finite dimensional approximation of equation (3.2); thus there exists a globally defined Euler flow, say it $U^{L,R}$, defined on $H^\beta_{loc}(\mathbb{R}^2)$.*

Proof. We study the following system of ODEs for all $k \in \mathbb{Z}^2$ with $k > 0$, $k_1 < LR_1$ and $k_2 < LR_2$:

$$\begin{aligned}\frac{d}{dt} U_k^{L,R}(t, \varphi^{L,R}) &= B_k^{L,R}(U^{L,R}(t, \varphi^{L,R})) \\ U_k^{L,R}(0, \varphi^{L,R}) &= \varphi_k^{L,R}\end{aligned}$$

for

$$\varphi^{L,R}(t,x) = \sum_{\substack{k>0 \\ k_1<LR_1 \\ k_2<LR_2}} \varphi_k^{L,R}(t) e_k^L(x) \in \mathbb{R}^d,$$

where $d = d(R) := \#\{k \in \mathbb{Z}^2 : k > 0 \text{ and } k_i < LR_i \text{ for } i = 1,2\}$ and where

$$B_k^{L,R}(\varphi^{L,R}) = \frac{1}{L}\left(\frac{2\pi}{L}\right)^2 \sum_{\substack{h>0 \\ h\neq k \\ h_1<LR_1 \\ h_2<LR_2}} \left[\frac{(h^\perp \cdot k)(k \cdot h)}{k^2} - \frac{h^\perp \cdot k}{2}\right] \varphi_h^{L,R} \varphi_{k-h}^{L,R}.$$

From the regularity of the finite dimensional quadratic vector field $B^{L,R}$ we know that there exists an associated global flow, that is for all positive $k \in \mathbb{Z}^2$ with $k_1 < LR_1$ and $k_2 < LR_2$ we have

$$U_k^{L,R}(t, \varphi^{L,R}) = \varphi_k^{L,R} + \int_0^t B_k^{L,R}(U^{L,R}(s, \varphi^{L,R}))ds, \qquad \forall\, t \in \mathbb{R}.$$

Now, for $\varphi^L \in H^\beta$ we write

$$\varphi^L = \Pi_R \varphi^L + \Pi_R^\perp \varphi^L = \varphi^{L,R} + \Pi_R^\perp \varphi^L,$$

where Π_R is the orthogonal projection on the subspace spanned by $\{e_k : k > 0 \text{ and } k_i < LR_i \text{ for } i = 1,2\}$. Therefore, if we define

$$U_k^{L,R}(t, \varphi^L) := U_k^{L,R}(t, \varphi^{L,R}) + \Pi_R^\perp \varphi^L,$$

then $U^{L,R}(t, \varphi^L)$ is in fact a $B^{L,R}$-flow on $H^\beta([0,L]^2)$. Finally, for $\varphi \in H^\beta_{loc}(\mathbb{R}^2)$ we write

$$\varphi = \varphi|_{[0,L]^2} + \varphi|_{[0,L]^{2C}} = \varphi^L + \varphi|_{[0,L]^{2C}}$$

and we define

$$U_k^{L,R}(t, \varphi) := U_k^{L,R}(t, \varphi^L) + \varphi|_{[0,L]^{2C}}\,;$$

it follows that $U^{L,R}(t, \varphi)$ is in fact a $B^{L,R}$-flow on $H^\beta_{loc}(\mathbb{R}^2)$. From the conservation of the energy we know that the flow is defined for all times. Furthermore we have

$$U^{L,R}(t, \varphi) = \sum_{\substack{k>0 \\ k_1<LR_1 \\ k_2<LR_2}} U_k^{L,R}(t, \varphi) e_k^L$$

with $U_k^{L,R}(\cdot, \varphi) \in C(\mathbb{R}; \mathbb{C})$ for all k. $\square$

3.4.2 Existence of an invariant flow

Here we prove the existence of an invariant flow for (3.2) taking values in $H^{\beta}_{loc}(\mathbb{R}^2)$ for $\beta < -1$.

Theorem 3.4.1. *Let $\beta < -1$. There exists a probability space $(\Omega, \mathcal{F}, P)$ and globally defined flow $U(\cdot, \omega) \in C(\mathbb{R}; H^{\beta}_{loc}(\mathbb{R}^2))$ for P- a.e. $\omega \in \Omega$, such that*

1. $$U(t,\omega) = U(0,\omega) + \int_0^t B(U(s,\omega))ds, \qquad P-a.e.\,\omega, \ \forall\, t \in \mathbb{R},$$

2. *the measure μ_γ is invariant under the flow, in the sense that*

$$\int f(U(t,\omega))dP(\omega) = \int f(\varphi)d\mu_\gamma(\varphi), \qquad \forall f \in C_b, \ \ \forall\, t \in \mathbb{R}.$$

Proof. From Proposition 3.3.1, we know that $\mu^R_{L,\gamma}$ is a weakly convergent sequence of probability measures in $H^{\beta}_{loc}(\mathbb{R}^2)$. Therefore, by Skorohod's theorem there exists a probability space $(\Omega, \mathcal{F}, P)$ and two stochastic processes $U^{L,R}, U$ with laws respectively $\mu^R_{L,\gamma}, \mu_\gamma$, such that $U^{L,R}(t,\omega)$ converges to $U(t,\omega)$ P- a.e. ω and for all $t \in \mathbb{R}$, when L, R tend to infinity. In particular, it follows that

$$\int f(U(t,\omega))dP(\omega) = \int f(\varphi)d\mu_\gamma(\varphi), \qquad \forall f \in C_b \ \ \forall\, t \in \mathbb{R}. \tag{3.19}$$

Moreover for all $L \in \mathbb{N}^*$ and for $\beta < -1$, we have

$$\int \sum_k \left(\frac{2\pi k}{L}\right)^{2\beta} |U^L_k(t,\omega)|^2 dP(\omega) = \int \|\varphi^L\|^2_{H^\beta} d\mu_{L,\gamma}(\varphi^L) \leq C < +\infty.$$

This implies that, $P-a.e.\,\omega$ and for all times, $U(t,\omega)$ takes values in $H^{\beta}_{loc}(\mathbb{R}^2)$.

Now we have to check that

$$\mathbb{E}_P d_{\beta,2}(\int_0^t [B^{L,R}_k(U^{L,R}(s,\omega)) - B_k(U(s,\omega))]ds; 0)$$

tends to 0 when L and R tend to infinity. We have

$$\begin{aligned}\mathbb{E}_P d_{\beta,2}(\int_0^t [B^{L,R}_k(U^{L,R}(s,\omega)) - B_k(U(s,\omega))]ds; 0) \leq \\ \mathbb{E}_P d_{\beta,2}(\int_0^t [B^{L,R}_k(U^{L,R}(s,\omega)) - B_k(U^{L,R}(s,\omega))]ds; 0) \\ + \mathbb{E}_P d_{\beta,2}(\int_0^t [B_k(U^{L,R}(s,\omega)) - B_k(U(s,\omega))]ds; 0).\end{aligned}$$

The first term is bounded by

$$\sum_{L\in\mathbb{N}^*} 2^{-L} \sum_k \left(\frac{2\pi k}{L}\right)^{2\beta} \int_0^t \mathbb{E}_P |B_k^{L,R}(U^{L,R}(s,\omega)) - B_k(U^{L,R}(s,\omega))|^2 ds.$$

It converges to 0 when L and R tend to infinity by the invariance of the measure and the L^2 convergence of $B_k^{L,R}$ towards B_k. Analogously the second term is bounded by

$$\sum_{L\in\mathbb{N}^*} 2^{-L} \sum_k \left(\frac{2\pi k}{L}\right)^{2\beta} \int_0^t \mathbb{E}_P |B_k(U^{L,R}(s,\omega)) - B_k(U(s,\omega))|^2 ds.$$

This term also converges to 0 when L and R go to infinity by the equi-integrability of the functions $B_k(U^{L,R}(s,\omega))$ and the convergence of the flows $U^{L,R}(s,\omega)$ towards $U(s,\omega)$ (similar to the arguments used in [3]).

It only remains us to prove that for every fixed initial data $\varphi \in H^\beta_{loc}(\mathbb{R}^2)$, $U(\cdot,\omega)$ is a continuous function of time in $H^\beta_{loc}(\mathbb{R}^2)$ for P-a.e. $\omega \in \Omega$. Let $t > t' \in \mathbb{R}$ be such that $|t - t'| < \delta$ for some $\delta > 0$. From the invariance property and Proposition 3.4.2 we have

$$\begin{aligned}
\mathbb{E}_P \sup_{|t-t'|<\delta} d_{\beta,2}(U(t,\omega);U(t',\omega)) &= \mathbb{E}_P \sup_{|t-t'|<\delta} \sum_L 2^{-L} \frac{\|D^\beta \int_{t'}^t B(U(s,\omega))ds\|_{L^2([0,L]^2)}}{1 + \|D^\beta \int_{t'}^t B(U(s,\omega))ds\|_{L^2([0,L]^2)}} \\
&\le \delta \sum_L 2^{-L} \mathbb{E}_P \|D^\beta B(U(s,\omega))\|_{L^2([0,L]^2)} \\
&= \delta \sum_L 2^{-L} \mathbb{E}_{\mu_\gamma} \|D^\beta B(\varphi)\|_{L^2([0,L]^2)} \underset{\delta\to 0}{\longrightarrow} 0.
\end{aligned}$$

□

3.4.3 Continuity

The flow is P- almost everywhere continuous from $H^\beta_{loc}(\mathbb{R}^2)$ to $H^\beta_{loc}(\mathbb{R}^2)$ and for all $t \in \mathbb{R}$. We write

$$\begin{aligned}
\mathbb{E}_P d_{\beta,2}(U_{\varphi_1}(t,\omega);U_{\varphi_2}(t,\omega)) &\le \mathbb{E}_P d_{\beta,2}(U_{\varphi_1}(t,\omega);U^n_{\varphi_1}(t,\omega)) \\
&\quad + \mathbb{E}_P d_{\beta,2}(U^n_{\varphi_1}(t,\omega);U^n_{\varphi_2}(t,\omega)) \\
&\quad + \mathbb{E}_P d_{\beta,2}(U^n_{\varphi_2}(t,\omega);U_{\varphi_2}(t,\omega))
\end{aligned}$$

where U^n denotes a finite dimensional approximation of U. On one hand there exist $n_1, n_2 \in \mathbb{N}$ such that for every $n \ge \max\{n_1, n_2\}$

$$\mathbb{E}_P d_{\beta,2}(U_{\varphi_1}(t,\omega);U^n_{\varphi_1}(t,\omega)) \le \frac{\varepsilon}{3} \quad \text{and} \quad \mathbb{E}_P d_{\beta,2}(U^n_{\varphi_2}(t,\omega);U_{\varphi_2}(t,\omega)) \le \frac{\varepsilon}{3}.$$

On the other hand, for a fixed $n \geq \max\{n_1, n_2\}$, U^n is continuous; indeed it is the flow associated with the quadratic vector field B^n. Thus there exists a positive δ such that for $d_{\beta,2}(\varphi_1; \varphi_2) \leq \delta$ we have

$$\mathbb{E}_P d_{\beta,2}(U^n_{\varphi_1}(t,\omega); U^n_{\varphi_2}(t,\omega)) \leq \frac{\varepsilon}{3}.$$

Chapter 4

On a non-periodic modified Euler equation: well-posedness and quasi-invariant measures

4.1 Introduction

Cauchy problem for the Euler equation is a challenging one in nonlinear partial differential equations. Local existence of smooth solutions was proved by Lichtenstein in 1925 [50]. In two-dimensions and in bounded domains existence, uniqueness and global regularity were shown for bounded initial vorticity by Judovic (1963) [46]. Solutions with initial data of finite energy were studied also by Kato [48] and Bardos [14], among others. There is an extensive literature about local solutions of Euler equations, but much less is known about global ones. The only known results to the authors are due to DiPerna and Majda [36] concerning a particular type of very weak solutions and a recent work [39] dealing with special function spaces which allow for unbounded vorticities.

The least action principle on the diffeomorphisms group (Arnold [12], Ebin-Marsden [37], more recently Brenier [19]) provides a different approach, that studies the Lagrangian problem for the position and not directly the Cauchy problem for the velocity field.

There is also the statistical approach to this type of equations, that consists in defining a priori invariant (or quasi-invariant) measures for the flow and using such measures to prove existence starting (almost everywhere) in the support of the measures. These supports are in general spaces of distributions. With respect to this approach, we mention [3] for the case of the periodic two-dimensional Euler equation. Recently, in [34] we have obtained by these methods local solutions in the plane.

In this work, we consider a modification of the Euler equation involving the pressure term, namely

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla)\tilde{u} = -\nabla p + cxp, \qquad \mathrm{div}\tilde{u} = 0,$$

with c a positive constant, which allows us to use the Ornstein-Ulhenbeck operator instead of the Laplacian in the vorticity equation and to use Sobolev spaces with respect to Gaussian measures. In order to give a sound to this equation we first look for weak solutions starting with bounded functions (Theorem 4.3.1). Then we construct quasi-invariant Gibbs-type measures and define global solutions of the equation for less regular initial conditions (in the support of such probability measures).

We use the exponential integrability conditions (Equation (4.34)), given in [30] and improved in [70] to prove the quasi-invariance property and existence of a unique statistical solution (Theorem 4.6.1) in a concrete example for the first time, up to our knowledge.

Moreover, thanks to some "dispersive bounds" for Hermite functions, firstly proved in dimension one by N. Burq, L. Thomann and N. Tzvetkov in [21] and successively extended to other dimensions by A. Poiret in his Ph.D. thesis [58], we show that the supports of such quasi-invariant measures actually contain more regular functions, namely $L^p_{loc}(\mathbb{R}^2)$ for $p \in (2, \frac{10}{3})$ (Theorem 4.4.2) and not only distributions.

Last we remark that we are considering a "regularizing" approximation of the Euler vector field, that in particular takes values in the Cameron-Martin space $H^2_{\sigma^c}(\mathbb{R}^2)$ (Proposition 4.5.1), however we cannot rigorously consider the limit of the approximations in view of Remark 4.6.1 below.

Organisation of the paper. In Section 2, we present a modification of the Euler equation and we recall some properties of Hermite polynomials and Gaussian Sobolev spaces. Existence and uniqueness of a weak solution for the corresponding vorticity equation is proved for bounded initial data in Section 3 (Theorem 4.3.1). In Section 4, we define Gibbs-type measures and, thanks to what was called the "dispersive bounds", we show that, in particular, the spaces $L^p_{loc}(\mathbb{R}^2)$ for $p \in (2, \frac{10}{3})$ belong to the supports of these measures (Theorem 4.4.2). In Section 5, we study the regularity of the vector field, its derivatives and its divergence. We need these technical results to prove, in Section 6, existence of an almost surely unique flow and the quasi-invariance property (Theorem 4.6.1).

4.2 The modified Euler equation

Let us consider the following modification of the Euler equation

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla)\tilde{u} = -\nabla p + cxp, \qquad \mathrm{div}\tilde{u} = 0 \tag{4.1}$$

where $\tilde{u} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ denotes the time dependent velocity field, $p : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ may depend on c and c is a fixed parameter in $(0, 1)$.

After the change of variables

$$u(t,x) = \sigma^c(x)\tilde{u}(t,x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^2 \tag{4.2}$$

where $\sigma^c(x) = \frac{\sqrt{c}}{2\pi} e^{-\frac{c|x|^2}{2}}$ denotes a Gaussian density in $\mathbb{R}^2$, the equation reads,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)(\rho^c u) = -\nabla(\sigma^c p), \qquad \mathrm{div}_{\rho^c} u = 0, \tag{4.3}$$

where $\rho^c(x) := (\sigma^c)^{-1}(x) = \frac{2\pi}{\sqrt{c}} e^{\frac{c|x|^2}{2}}$ and $\mathrm{div}_{\rho^c} u$ is defined by

$$\int_{\mathbb{R}^2} \mathrm{div}_{\rho^c} u f d\rho^c = -\int_{\mathbb{R}^2} u \cdot \nabla f d\rho^c, \qquad \forall f \in \mathcal{C}_c^1$$

(for simplicity, we use the notation $d\rho^c = \rho^c dx$). We assume that the initial condition for (4.3) is defined by $u_0 = \sigma^c \tilde{u}_0$, where $\tilde{u}_0$ is the initial data for (4.1), and that $\tilde{u}$ and u vanish sufficiently rapidly at infinity.

As we will see below, this change of variables is convenient to study the equations in $L^2_{\sigma^c}(\mathbb{R}^2)$, the space of real-valued functions that are square integrable with respect to the measure $\sigma^c dx$.

Hermite polynomials and Gaussian Sobolev spaces

We recall the definition of the k-th order Hermite polynomial on $\mathbb{R}^2$

$$H_k^c(x) := \Pi_{i=1,2} H_{k_i}^c(x_i), \qquad k \in \mathbb{Z}^2,\ k \geq 0$$

where

$$H_{k_i}^c(x_i) = \frac{1}{c^{1/4}\sqrt{c^{k_i}}\sqrt{k_i!}} e^{\frac{cx_i^2}{2}} \frac{\partial^{k_i}}{\partial x_i^{k_i}} e^{-\frac{cx_i^2}{2}}, \qquad i = 1,2$$

denotes the one-dimensional Hermite polynomial of order k_i. We write

$$H_k^c(x) = \frac{1}{\sqrt{c}\sqrt{c^{|k|}}\sqrt{k!}} e^{\frac{c|x|^2}{2}} D^k e^{-\frac{c|x|^2}{2}}$$

where $|k| = k_1 + k_2$, $k! = k_1!k_2!$ and $D^k = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}}$. It is well known that the collection $\{H_k^c(x)\}_{k\geq 0}$ forms an orthonormal basis for $L^2_{\sigma^c}(\mathbb{R}^2)$. Moreover, the Hermite polynomials are eigenfunctions for the Ornstein-Uhlenbeck operator, $L^c : L^2_{\sigma^c}(\mathbb{R}^2) \to L^2_{\sigma^c}(\mathbb{R}^2)$, defined by

$$L^c \varphi = \Delta\varphi - cx \cdot \nabla\varphi.$$

We have

$$L^c H_k^c(x) = -c|k| H_k^c(x), \qquad \forall\ k \geq 0.$$

We recall some properties of the Hermite polynomials that we will use below. For the one-dimensional Hermite polynomials we have

1. *Differentiation formula*:
$$\frac{d}{dx} H_n^c(x) = -\sqrt{nc} H_{n-1}^c(x); \tag{4.4}$$

2. *Recursive relation*:
$$\sqrt{n+1} H_{n+1}^c(x) + \sqrt{c} x H_n^c(x) + \sqrt{n} H_{n-1}^c(x) = 0; \tag{4.5}$$

3. *Product formula*:
$$H_n^c(x) H_m^c(x) = \frac{1}{c^{\frac{1}{4}}} \sum_{r \le n \wedge m} \Theta(n,m,r) H_{n+m-2r}^c(x), \tag{4.6}$$

   where
$$\Theta(n,m,r) = \left[\binom{n}{r} \binom{m}{r} \binom{n+m-2r}{n-r} \right]^{1/2}. \tag{4.7}$$

Remark 4.2.1. The first property is well-known, see for example [54]. The product formula can be found in [38] (pag. 195, equation (37)); where the formula is stated for the "physicists" Hermite polynomials, $\mathcal{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. From the relation $H_n^c(x) = \frac{2^{-\frac{n}{2}}(-1)^n}{c^{\frac{1}{4}}\sqrt{n!}} \mathcal{H}_n(\sqrt{\frac{c}{2}} x)$ we get equations (4.6)-(4.7).

Properties (4.4) to (4.7) can be generalised for the two-dimensional Hermite polynomials. If $k \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$, we have

1. *2D Differentiation formula*:
$$\nabla H_k^c(x) = -\sqrt{c} \left(\sqrt{k_1} H_{k_1-1}^c(x_1) H_{k_2}^c(x_2), \sqrt{k_2} H_{k_1}^c(x_1) H_{k_2-1}^c(x_2) \right); \tag{4.8}$$

2. *2D Recursive relation*:

   for $i = 1,2$ and $j \neq i$,
$$\sqrt{k_i+1} H_{k_i+1}^c(x_i) H_{k_j}^c(x_j) + \sqrt{c} x_i H_{k_i}^c(x_i) H_{k_j}^c(x_j) + \sqrt{k_i} H_{k_i-1}^c(x_i) H_{k_j}^c(x_j) = 0; \tag{4.9}$$

3. *2D Product formula*:
$$\begin{aligned} H_k^c(x) H_h^c(x) &= \frac{1}{\sqrt{c}} \sum_{\substack{r_1 \le k_1 \wedge h_1 \\ r_2 \le k_2 \wedge h_2}} \Theta(k_1,h_1,r_1) \Theta(k_2,h_2,r_2) H_{k_1+h_1-2r_1}^c(x_1) H_{k_2+h_2-2r_2}^c(x_2) \\ &= \frac{1}{\sqrt{c}} \sum_{r \le k \wedge h} \tilde{\Theta}(k,h,r) H_{k+h-2r}^c(x), \end{aligned} \tag{4.10}$$

where
$$\tilde{\Theta}(k,h,r) := \Pi_{i=1,2}\Theta(k_i,h_i,r_i)$$
and Θ is defined in (4.7).

Also consider for all $\beta \in \mathbb{R}$ the function spaces
$$\mathcal{H}^{\beta}_{\sigma^c}(\mathbb{R}^2) = \left\{ v : \mathbb{R}^2 \to \mathbb{R}, v \in L^2_{\sigma^c}(\mathbb{R}^2) \ : \ (I - L^c)^{\beta/2} v \in L^2_{\sigma^c}(\mathbb{R}^2) \right\};$$
for β negative or non-integer the operator L^c is understood as a pseudo-differential operator in the Gaussian space of square integrable functions. The Sobolev spaces $\mathcal{H}^{\beta}_{\sigma^c}(\mathbb{R}^2)$ may be identified with the complex spaces
$$H^{\beta}_{\sigma^c}(\mathbb{R}^2) = \left\{ v = \sum_{k \geq 0} v_k H^c_k \ : \ \sum_{k\geq 0}(1 + c|k|)^{\beta}|u_k|^2 < +\infty \right\}.$$
These are Hilbert spaces with inner products given by
$$< u, v >_{\beta,\sigma^c} = \sum_{k\geq 0}(1 + c|k|)^{\beta} u_k \bar{v}_k.$$
By $\|\cdot\|_{\beta,\sigma^c}$ we denote the norm of $H^{\beta}_{\sigma^c}(\mathbb{R}^2)$ for all $\beta \in \mathbb{R}$.

4.3 The vorticity equation

As usual the vorticity equations are obtained by taking the “curl” of equation (4.3). We have
$$\nabla^{\perp} \cdot [(u \cdot \nabla)(\rho^c u)] = \nabla^{\perp} \cdot [(\sigma^c \tilde{u} \cdot \nabla)\tilde{u}] = \sum_{i,j=1,2} \partial^{\perp}_i \sigma^c \tilde{u}_j \partial_j \tilde{u}_i + \sigma^c \partial^{\perp}_i \tilde{u}_j \partial_j \tilde{u}_i + \sigma^c \tilde{u}_j \partial^{\perp}_i \partial_j \tilde{u}_i,$$
where
$$\sum_{i,j=1,2} \partial^{\perp}_i \sigma^c \tilde{u}_j \partial_j \tilde{u}_i = \sum_{i,j=1,2} -c x^{\perp}_i \sigma^c \tilde{u}_j \partial_j \tilde{u}_i = \sum_{i,j=1,2} -\sigma^c \tilde{u}_j \partial_j (c x^{\perp}_i \tilde{u}_i)$$
and
$$\sum_{i,j=1,2} \sigma^c \partial^{\perp}_i \tilde{u}_j \partial_j \tilde{u}_i = \sum_{i,j=1,2} \sigma^c \partial_i \tilde{u}_i \partial^{\perp}_j \tilde{u}_j = 0,$$
since $\mathrm{div}\tilde{u} = \sum_{i=1,2} \partial_i \tilde{u}_i = 0$. Also we have
$$\nabla^{\perp} \cdot \nabla(\sigma^c p) = 0.$$
Moreover, since $\mathrm{div}\tilde{u} = 0$ we known that there exists a real-valued function $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ such that $\tilde{u} = \nabla^{\perp}\varphi$ and $u = \sigma^c \nabla^{\perp}\varphi$. Thus we have
$$\nabla^{\perp} \cdot \tilde{u} = \sigma^c \Delta\varphi \quad \text{ and } \quad \nabla^{\perp} \cdot u = \sigma^c L^c \varphi,$$

from which follows that the vorticity equation can be written as

$$\frac{\partial}{\partial t}\sigma^c L^c\varphi = -(\sigma^c\nabla^\perp\varphi\cdot\nabla)L^c\varphi, \tag{4.11}$$

or equivalently as

$$\frac{\partial}{\partial t}L^c\varphi = -(\nabla^\perp\varphi\cdot\nabla)L^c\varphi. \tag{4.12}$$

In particular, we observe that the quantity $L^c\varphi$ is conserved along the particle trajectories with velocity $\tilde{u}$, that we denote by Φ_t, that is

$$L^c\varphi(t,x) = L^c\varphi(0,\Phi_{-t}(x)), \qquad t\in\mathbb{R},\ x\in\mathbb{R}^2. \tag{4.13}$$

Indeed, by definition of particle trajectories, we have

$$\begin{aligned}\frac{d}{dt}\Phi_t(x) &= \rho^c u(\Phi_t(x),t)\\ \Phi_0(x) &= x,\end{aligned} \tag{4.14}$$

thus for all $x\in\mathbb{R}^2$

$$\begin{aligned}\frac{d}{dt}L^c\varphi(t,\Phi_t(x)) &= \frac{\partial}{\partial_t}L^c\varphi(t,\Phi_t(x)) + \frac{d}{dt}\Phi_t(x)\cdot\nabla L^c\varphi(t,\Phi_t(x))\\ &= \frac{\partial}{\partial_t}L^c\varphi(t,\Phi_t(x)) + \rho^c u(t,\Phi_t(x))\cdot\nabla L^c\varphi(t,\Phi_t(x)) = 0,\end{aligned}$$

where the last equality follows from (4.12).

The L^p-norms of $L^c\varphi$ are conserved for all $p\in\{1,2,\dots,\infty\}$; indeed for any f measurable function

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} f(L^c\varphi(t,\Phi_t(x)))dx = \int_{\mathbb{R}^2} f'(L^c\varphi(t,\Phi_t(x)))[&\frac{\partial}{\partial_t}L^c\varphi(t,\Phi_t(x))\\ &+ \rho^c u(t,\Phi_t(x))\cdot\nabla L^c\varphi(t,\Phi_t(x))]dx = 0.\end{aligned}$$

For $p=2$, we directly prove the statement

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|L^c\varphi\|_{L^2}^2 = -\int(\nabla^\perp\varphi\cdot\nabla)L^c\varphi L^c\varphi dx &= \int \operatorname{div}(\nabla^\perp\varphi L^c\varphi)L^c\varphi dx\\ &= \int(\nabla^\perp\varphi\cdot\nabla)L^c\varphi L^c\varphi dx = 0.\end{aligned}$$

Existence and uniqueness

In this section we look for weak solutions of equations (4.3). By equation (4.13), we obtain weak solutions of

$$\frac{\partial}{\partial t}L^c\varphi = -(\nabla^\perp\varphi\cdot\nabla)L^c\varphi,$$

if we are able to solve the associated ODE for the particle trajectories

$$\begin{aligned}\frac{d}{dt}\Phi_t(x) &= \rho^c u(\Phi_t(x),t)\\ \Phi_0(x) &= x.\end{aligned}$$

If we define by ω the vorticity of u, that is

$$\omega = \nabla^\perp \cdot u = \sigma^c L^c \varphi,$$

we have

$$\rho^c u = K_{L^c} * \rho^c \omega,$$

where $K_{L^c}(x,y)$ denotes the orthogonal gradient of G_{L^c}, that in turn denotes the Green's function for the Ornstein-Uhlenbeck operator L^c on $\mathbb{R}^2$. We consider initial data $\rho^c\omega_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$.

In order to compute G_{L^c}, we consider the operator $L^c = \Delta - cx\cdot\nabla$ as a perturbation of Δ, thus we write G_{L^c} in terms of G, where G denotes the Green's function for Δ in $\mathbb{R}^2$. It is well-known that $G(x,y) = \frac{1}{2\pi}\ln|x-y|$. By definition

$$L^c G_{L^c}(x,y) = \delta(x-y) = \Delta G(x,y)$$

in the sense of distributions, that is

$$\forall f\in L^\infty,\quad \int L^c G_{L^c}(x,y)f(y)dy = \int \delta(x-y)f(y)dy = \int \Delta G(x,y)f(y)dy.$$

Now the idea is to apply Δ^{-1} to both members of the latter expression. Since $\rho^c u = K_{L^c} * \rho^c\omega$, we use Δ^{-1} in $L^1_{\rho^c}$. It is easy to check that

$$G^{\rho^c}(x,y) = \sigma^c(y)G(x,y) \tag{4.15}$$

and

$$G^{\rho^c}_{L^c}(x,y) = \sigma^c(y)G_{L^c}(x,y) \tag{4.16}$$

where G^{ρ^c} and $G^{\rho^c}_{L^c}$ denote respectively the Green's functions for Δ and L^c in $L^1_{\rho^c}$. Hence, we get

$$G^{\rho^c}_{L^c} = G^{\rho^c} + G^{\rho^c} * x\cdot\nabla_x G^{\rho^c}_{L^c} \tag{4.17}$$

where this should be understood as

$$\forall f\in L^\infty,\quad \int G^{\rho^c}_{L^c}(x,y)f(y)dy = \int G^{\rho^c}(x,y)f(y)dy + \int\int G^{\rho^c}(x,z)z\cdot\nabla_z G^{\rho^c}_{L^c}(z,y)f(y)dzdy.$$

Using iteratively equation (4.17), we get an expression for $\rho^c u = K_{L^c} * \rho^c \omega$ such that

$$|\rho^c(x)u(x,t)| = |\nabla^\perp \varphi(x,t)| \le \|\rho^c\omega\|_{L^\infty} \left\{ \frac{1}{2\pi} \int \frac{d\sigma^c(y)}{|x-y|} + \left(\frac{1}{2\pi}\right)^2 \int\int \frac{c|x_1| d\sigma^c(x_1) d\sigma^c(y)}{|x-x_1||x_1-y|} + \cdots + \left(\frac{1}{2\pi}\right)^n \int \cdots \int \frac{c^{n-1}|x_1|\cdots|x_{n-1}| d\sigma^c(x_1)\cdots d\sigma^c(x_{n-1}) d\sigma^c(y)}{|x-x_1|\cdots|x_{n-1}-y|} + \dots \right\}.$$

The n-th term of the previous expansion is smaller than $\frac{\sqrt{2\pi}}{2}\left(\frac{1}{2\pi}\right)^n c^{n-1}$, thus we have $|\rho^c u| \lesssim \|\rho^c\omega\|_{L^\infty}$, where $\lesssim$ stands for less or equal up to a multiplicative constant.

We prove the following result about existence of weak solutions for the vorticity equation, showing that the problem is well-posed. We have not found in the literature a result from which ours could be directly deduced.

Theorem 4.3.1. *Given $\rho^c\omega_0 \in L^1 \cap L^\infty$, there exists $T > 0$ such that equation (4.14) has a unique solution in $[-T,T]$ and $\rho^c\omega \in L^\infty([-T,T]; L^1 \cap L^\infty)$ is a weak solution for equation (4.12).*

Proof. By Osgood's theorem in Banach spaces (see [62]), if $\rho^c u$ is a quasi-Lipschitz field, we obtain a unique solution for the Cauchy problem (4.14) in $[-T,T]$. For $x, x' \in \mathbb{R}^2$, we have

$$|\rho^c(x)u(x,t) - \rho^c(x')u(x',t)| \le \|\rho^c\omega\|_{L^\infty} \left\{ \frac{1}{2\pi} \int \left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| d\sigma^c(y) + \left(\frac{1}{2\pi}\right)^2 \int\int \left| \frac{(x-x_1)^\perp}{|x-x_1|^2} - \frac{(x'-x_1)^\perp}{|x'-x_1|^2} \right| \frac{c|x_1| d\sigma^c(x_1) d\sigma^c(y)}{|x_1-y|} + \dots \right\}.$$

Below we follow Appendix 2.3 of [55] (where the case of a bounded domain is treated) to prove the quasi-Lipschitz continuity. Let $r := |x-x'|$; for $r \ge 1$ the statement is a consequence of the previous computations, for $r < 1$ we set $A := \{y \in \mathbb{R}^2 \mid |x-y| \le 2r\}$ and we write

$$\int_{\mathbb{R}^2} \left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| d\sigma^c(y) = \int_A \left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| d\sigma^c(y) + \int_{A^c} \left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| d\sigma^c(y).$$

On one hand,

$$\int_A \left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| d\sigma^c(y) \le \int_{|x-y|\le 2r} \left[\frac{1}{|x-y|} + \frac{1}{|x'-y|} \right] d\sigma^c(y) \lesssim r.$$

On the other, choosing x'' to be a point belonging to the segment x, x', for $y \in A^c$ we have $|x'' - y| \geq \frac{1}{2}|x - y|$, thus

$$\begin{aligned}\int_{A^c} \left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| d\sigma^c(y) &\lesssim r \int_{A^c} \frac{1}{|x''-y|^2} d\sigma^c(y) \\ &\lesssim r \left\{ \int_{2r<|x-y|<2} \frac{1}{|x-y|^2} d\sigma^c(y) + \int_{\mathbb{R}^2} d\sigma^c(y) \right\} \\ &\lesssim r \left\{ \int_{2r<|x-y|<2} \frac{1}{|x-y|^2} dy + 1 \right\}.\end{aligned}$$

Computing the integrals we obtain

$$\int_{\mathbb{R}^2} \left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| d\sigma^c(y) \lesssim \lambda(|x - x'|)$$

where λ, defined by $\lambda(r) = r$, for $r \geq 1$ and by $\lambda(r) = r(1 - \ln r)$, for $r < 1$, is the modulus of continuity for $\rho^c u$. That is $\rho^c u$ is quasi-Lipschitz continuous and by Osgood's theorem there exists a unique flow given by $\Phi_t(x) = x + \int_0^t \rho^c(x) u(s, \Phi_s(x)) ds$ for $t \in [-T, T]$. From $\rho^c(x)\omega(x,t) = \rho^c(\Phi_{-t}(x))\omega_0(\Phi_{-t}(x))$ and the assumptions we get $\rho^c\omega \in L^\infty([-T,T]; L^1 \cap L^\infty)$, which is sufficient to verify the vorticity equation in the weak sense, that is

$$\frac{d}{dt} \int \rho^c \omega f dx = \int \rho^c \omega (\nabla^\perp \varphi \cdot \nabla f) dx, \qquad \forall f \in C_0^1.$$

□

4.4 Quasi-invariant measures

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we let $\{g_k\}_{k \in \mathbb{Z}^2}$ to be a sequence of independent and identically distributed random variables, where each g_k is distributed as a standard, complex-valued Gaussian. We denote by $\tilde{\lambda}_k$ the eigenvalues of the Ornstein-Uhlenbeck operator L^c on $L^2_{\sigma^c}$, that is $\tilde{\lambda}_k = -c|k|$ for all $k \geq 0$. For any given n, we consider the random variable

$$\Gamma^n_\gamma(\omega, x) := \sum_{k \in \{\alpha_1, \ldots, \alpha_{d(n)}\}} \frac{g_k(\omega)}{1 - \tilde{\lambda}_k} H^c_k(x),$$

whose law is given by

$$d\mu^n_{\sigma^c,\gamma}(\varphi) \simeq \prod_{k \in \{\alpha_1, \ldots, \alpha_{d(n)}\}} \frac{\gamma(1 + c|k|)^2}{2\pi} e^{-\frac{\gamma}{2}(1+c|k|)^2 |\varphi_k|^2} d\varphi_k,$$

for every $\varphi(t,x) = \sum_{k\geq 0}\varphi_k(t)H_k^c(x) \in L^2_{\sigma^c}$. The $\alpha_1,\ldots,\alpha_{d(n)}$ denote non-negative pairs of $\mathbb{Z}^2$. In the limit when n tends to infinity $\Gamma^n_\gamma(\omega,x)$ converges pointwise to $\Gamma_\gamma(\omega,x) := \sum_{k\geq 0}\frac{g_k(\omega)}{1-\tilde{\lambda}_k}H_k^c(x)$ and we denote by $d\mu_{\sigma^c,\gamma}$ its law. For $\varphi \in L^2_{\sigma^c}$,

$$\begin{aligned} d\mu_{\sigma^c,\gamma}(\varphi) &\simeq \prod_{k\geq 0}\frac{\gamma(1+c|k|)^2}{2\pi}e^{-\frac{\gamma}{2}(1+c|k|)^2|\varphi_k|^2}d\varphi_k \\ &\simeq \frac{1}{Z_\gamma}e^{-\frac{\gamma}{2}\|(I-L^c)\varphi\|^2_{L^2_{\sigma^c}}}\mathcal{D}\varphi, \end{aligned}$$

thus $\mu_{\sigma^c,\gamma}$ is formally the Gibbs-type measure associated to the quantity $\frac{1}{2}\|(I-L^c)\varphi\|^2_{L^2_{\sigma^c}}$.

For any $\gamma \in \mathbb{R}^+$, the triple $(H^{-\varepsilon}_{\sigma^c}, H^2_{\sigma^c}, d\mu_{\sigma^c,\gamma})$ is a complex abstract Wiener space for $\varepsilon > 0$; $H^{-\varepsilon}_{\sigma^c}$ is the support of $\mu_{\sigma^c,\gamma}$ and $H^2_{\sigma^c}$ is the Cameron-Martin space. In particular; $\mathbb{E}_{\mu_{\sigma^c,\gamma}}(\varphi_k\bar{\varphi}_h) = \delta_{k,h}\frac{2}{\gamma(1+c|k|)^2}$, $\mathbb{E}_{\mu_{\sigma^c,\gamma}}(\varphi_k) = 0$, and $\mathbb{E}_{\mu_{\sigma^c,\gamma}}|\varphi_k|^{2r} = \frac{2^r r!}{\gamma^r(1+c|k|)^{2r}}$.

Now we prove that the supports of the measures $\mu_{\sigma^c,\gamma}$ are not only spaces of very irregular functionals, but that in fact contain regular functions. Namely, $L^p_{loc}(\mathbb{R}^2) \subset \mathrm{supp}_{\mu_{\sigma^c,\gamma}}$ for every $p \in (2, 10/3)$. We will use the so called "dispersive bound" for Hermite functions, firstly proved in dimension one by N. Burq, L. Thomann and N. Tzvetkov in [21] and extended to other dimensions by A. Poiret in his Ph.D. thesis [58].

Below we denote by $h_k(x)$ the k-th order Hermite's function on $\mathbb{R}^2$, defined by $h_k(x) = h_{k_1}(x_1)\times h_{k_2}(x_2)$ for all $x\in\mathbb{R}^2$ and for all non-negative $k \in \mathbb{Z}^2$, where

$$h_{k_i}(x_i) = \frac{(-1)^n 2^{-k_i/2}}{\sqrt{\sqrt{\pi}k_i!}}\frac{d^{k_i}}{dx^{k_i}}(e^{-x_i^2})e^{x_i^2/2}, \qquad i=1,2.$$

It is well known that h_k is an eigenfunction, with corresponding eigenvalue denoted by λ_k^2, for the harmonic oscillator $H := -\Delta + |x|^2$ on $L^2(\mathbb{R}^2)$, that is $Hh_k = \lambda_k^2 h_k$. The eigenvalues are $\lambda_k^2 = \lambda_{k_1}^2 + \lambda_{k_2}^2 = (2k_1+1)+(2k_2+1) = 2(|k|+1)$. For further details see [58]. The relation between the Hermite's polynomials and the Hermite's functions is the following:

$$H_k^c(x) = \frac{(-1)^k}{\sqrt{c}}\sqrt{\sqrt{\pi}}\; h_k\left(\sqrt{\frac{c}{2}}x\right)e^{\frac{c|x|^2}{4}}. \tag{4.18}$$

The following result was proved in [58].

Theorem 4.4.1 (Dispersive bound). *Let $d\geq 2$. There exists a constant $C>0$ such that for all n,m*

$$\|h_n h_m\|_{L^2(\mathbb{R}^d)} \leq C\times\begin{cases}\max(\lambda_n,\lambda_m)^{-\frac{2}{3}+\frac{d}{6}}; & 2\leq d\leq 4\\ \max(\lambda_n,\lambda_m)^{-2+\frac{d}{2}}; & d\geq 4.\end{cases}$$

Moreover there exists a positive constant C such that for all n,m

$$\|h_n h_m\|_{L^{\frac{d+3}{d+1}}(\mathbb{R}^d)} \leq C\max(\lambda_n,\lambda_m)^{-\frac{1}{d+1}}.$$

In the two-dimensional case and for particular values of p, we show that the above result implies the following control over the L^p-norms of the Hermite's functions.

Corollary 4.4.1. *For all* $p \in (2, \frac{10}{3})$,

$$\forall\ n, \qquad \|h_n\|_{L^p(\mathbb{R}^2)} \leq C\lambda_n^{(\theta-1)/6},$$

where $\theta \in (0,1)$ *is such that* $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{10/3}$.

Proof. On one hand, from Theorem 4.4.1 when $d = 2$ and $n = m$, we get $\|h_n^2\|_{L^{5/3}} \leq C\lambda_n^{-1/3}$. This implies

$$\|h_n\|_{L^{10/3}(\mathbb{R}^2)} \leq C\lambda_n^{-1/6}. \tag{4.19}$$

On the other hand, by Hölder's inequality,

$$\|h_n\|_{L^p(\mathbb{R}^2)} \leq C\|h_n\|^{\theta}_{L^2(\mathbb{R}^2)}\|h_n\|^{1-\theta}_{L^{10/3}(\mathbb{R}^2)},$$

where $\theta \in (0,1)$ is such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{10/3}$. From the fact that $\{h_n\}_{n\geq 0}$ is an orthonormal basis for L^2 and from the bound (4.19) we conclude that

$$\|h_n\|_{L^p(\mathbb{R}^2)} \leq C\|h_n\|^{1-\theta}_{L^{10/3}(\mathbb{R}^2)} \leq C\lambda_n^{(\theta-1)/6}.$$

□

Below we translate the above bounds in terms of Hermite's polynomials.

Corollary 4.4.2. *For all* $p \in (2, \frac{10}{3})$,

$$\forall\ n, \qquad \|H_n^c\|_{L^p_{loc}(\mathbb{R}^2)} \leq C(p,c,R)\lambda_n^{(\theta-1)/6},$$

where $\theta \in (0,1)$ *is such that* $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{10/3}$ *and where* R *is a geometric constant that depends on each compact subset of* $\mathbb{R}^2$ *considered.*

Proof. Let $R > 0$; by the relation (4.18) we have

$$\begin{aligned}\|H_n^c\|_{L^p_{(\{|x|<R\})}} &= \frac{\pi^{1/4}}{\sqrt{c}}\left(\int_{\{|x|<R\}}\left|h_n\left(\sqrt{\frac{c}{2}}x\right)\right|^p e^{\frac{cp}{4}|x|^2}dx\right)^{1/p}\\ &\leq C(p,c)\left(\int_{\{|x|<\sqrt{\frac{c}{2}}R\}}|h_n(x)|^p\, e^{\frac{p}{2}|x|^2}dx\right)^{1/p}\\ &\leq C(p,c,R)\left(\int|h_n(x)|^p\,dx\right)^{1/p}.\end{aligned}$$

If $p \in (2, \frac{10}{3})$, by Corollary 4.4.1 and since R is arbitrary, we get

$$\|H_n^c\|_{L^p_{loc}(\mathbb{R}^2)} \leq C(p,c,R)\lambda_n^{(\theta-1)/6}.$$

□

Here we characterise the supports of the measures $\mu_{\sigma^c,\gamma}$ and in particular we see that they contain regular functions (and not only distributions).

Theorem 4.4.2. *Let $\varepsilon > 0$ and $p \in (2, \frac{10}{3})$; then*

$$\mathrm{supp}_{\mu_{\sigma^c,\gamma}} = H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2) \cap L^p_{loc}(\mathbb{R}^2).$$

Proof. As $\mu_{\sigma^c,\gamma}$ is the law of the random variable Γ_γ, its support is given by the spaces in which $\Gamma_\gamma(\omega,\cdot)$ takes values $\mathbb{P}$-almost surely. For any arbitrary $R > 0$ we have

$$\begin{aligned}
\left(\int_{\{|x|<R\}} \|\Gamma_\gamma(\omega,x)\|^p_{L^2_\omega} dx\right)^{1/p} &= \left\{\int_{\{|x|<R\}} \left[\mathbb{E}_{\mathbb{P}}\left(\sum_{h,k} \frac{g_k(\omega)\bar{g}_h(\omega)}{(1-\tilde{\lambda}_k)(1-\tilde{\lambda}_h)} H^c_k(x) H^c_h(x)\right)\right]^{p/2} dx\right\}^{1/p} \\
&= \left(\int_{\{|x|<R\}} \left|\sum_k \frac{|H^c_k(x)|^2}{(1-\tilde{\lambda}_k)^2}\right|^{p/2} dx)\right)^{1/p} \\
&= \left\|\sum_k \frac{|H^c_k(x)|^2}{(1-\tilde{\lambda}_k)^2}\right\|^{1/2}_{L^{p/2}(\{|x|<R\})} \\
&\le \left(\sum_k \frac{1}{(1-\tilde{\lambda}_k)^2} \|H^c_k\|^2_{L^p(\{|x|<R\})}\right)^{1/2},
\end{aligned}$$

which in turn, by Corollary 4.4.2,

$$\le C(p,c,R)\left(\sum_k \frac{\lambda_k^{-2\delta(k)}}{(1-\tilde{\lambda}_k)^2}\right)^{1/2} \le C(p,c,R)\left(\sum_k \frac{1}{(1+c|k|)^{2+\delta(k)}}\right)^{1/2} < +\infty$$

with $\delta(k)$ a strictly positive quantity. Moreover, for any $\varepsilon > 0$ we have

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}} \|\varphi\|^2_{-\varepsilon,\sigma^c} = \sum_k (1+c|k|)^{-\varepsilon} \mathbb{E}_{\mu_{\sigma^c,\gamma}} |\varphi_k|^2 = \frac{2}{\gamma} \sum_k \frac{1}{(1+c|k|)^{2+\varepsilon}} < +\infty.$$

□

4.5 The vorticity vector field

Similarly to what was previously done for Euler equation in a compact domain (c.f. [3]), we want to write the vorticity equation,

$$\partial_t L^c \varphi = -(\nabla^\perp \varphi \cdot \nabla) L^c \varphi,$$

as an infinite system of ordinary differential equations, using the orthonormal basis of $L^2_{\sigma^c}(\mathbb{R}^2)$ made of the Hermite polynomials $\{H^c_k\}_{k\in\mathbb{Z}^2}$. Let $\varphi \in L^2_{\sigma^c}(\mathbb{R}^2)$ be such that $\varphi(t,x) = \sum_{k\geq 0}\varphi_k(t)H^c_k(x)$ for some $\varphi_k : \mathbb{R} \to \mathbb{C}$ to determine. On one hand

$$\partial_t L^c\varphi(t,x) = -c\sum_{k\geq 0}|k|\frac{d}{dt}\varphi_k(t)H^c_k(x), \tag{4.20}$$

on the other

$$-(\nabla^\perp\varphi\cdot\nabla)L^c\varphi = c\sum_{p\geq 0}\sum_{\substack{q\geq 0\\ |q|<|p|}}(|p|-|q|)\varphi_p\varphi_q\nabla^\perp H^c_p\cdot\nabla H^c_q,$$

since $\nabla^\perp H^c_p\cdot\nabla H^c_q = -\nabla H^c_p\cdot\nabla^\perp H^c_q$. By $p \geq 0$ we mean $p_i \geq 0$, for $i = 1, 2$. From Hermite polynomial's properties (4.8) and (4.10) we have

$$\begin{aligned}
\nabla^\perp H^c_p\cdot\nabla H^c_q &= -c\sqrt{p_2q_1}H^c_{p_1}(x_1)H^c_{p_2-1}(x_2)H^c_{q_1-1}(x_1)H^c_{q_2}(x_2)\\
&\quad + c\sqrt{p_1q_2}H^c_{p_1-1}(x_1)H^c_{p_2}(x_2)H^c_{q_1}(x_1)H^c_{q_2-1}(x_2)\\
&= -\sqrt{c}\sqrt{p_2q_1}\sum_{\substack{r_1\leq p_1\wedge q_1-1\\ r_2\leq p_2-1\wedge q_2}}\Theta(p_1,q_1-1,r_1)\Theta(p_2-1,q_2,r_2)H^c_{p+q-1-2r}(x)\\
&\quad + \sqrt{c}\sqrt{p_1q_2}\sum_{\substack{r_1\leq p_1-1\wedge q_1\\ r_2\leq p_2\wedge q_2-1}}\Theta(p_1-1,q_1,r_1)\Theta(p_2,q_2-1,r_2)H^c_{p+q-1-2r}(x)\\
&= \sqrt{c}\sum_{|r|<|q|}[-\sqrt{p_2q_1}\Theta(p_1,q_1-1,r_1)\Theta(p_2-1,q_2,r_2)\\
&\quad +\sqrt{p_1q_2}\Theta(p_1-1,q_1,r_1)\Theta(p_2,q_2-1,r_2)]\,H^c_{p+q-1-2r}(x),
\end{aligned}$$

where in the last equality we used $|q| < |p|$. We define $k = p + q - 1 - 2r$, then $r = (p+q-1-k)/2$ and $0 < |k| < 2|p|$; we get

$$-(\nabla^\perp\varphi\cdot\nabla)L^c\varphi = c\sqrt{c}\sum_{p\geq 0}\sum_{0<|k|<2|p|}\sum_{\substack{q\geq 0\\ |q|<|p|}}(|p|-|q|)A(p,q,k)\varphi_p\varphi_q H^c_k(x), \tag{4.21}$$

where

$$\begin{aligned}
A(p,q,k) := [&-\sqrt{p_2q_1}\Theta(p_1,q_1-1,(p_1+q_1-1-k_1)/2)\Theta(p_2-1,q_2,(p_2+q_2-1-k_2)/2)\\
&+\sqrt{p_1q_2}\Theta(p_1-1,q_1,(p_1+q_1-1-k_1)/2)\Theta(p_2,q_2-1,(p_2+q_2-1-k_2)/2)]\,.
\end{aligned} \tag{4.22}$$

Comparing equations (4.20) and (4.21), the vector field B^c, corresponding to the equation

$$\frac{\partial}{\partial t}\varphi(t,x) = B^c(\varphi(t,x))$$

where φ denotes the Euler stream-function, can be written as follows

$$B^c(\varphi) = -\sqrt{c}\sum_{p\geq 0}\sum_{0<|k|<2|p|}\sum_{\substack{q\geq 0\\|q|<|p|}}\frac{1}{|k|}(|p|-|q|)A(p,q,k)\varphi_p\varphi_q H_k^c(x). \tag{4.23}$$

Remark 4.5.1 (Properties of A(p,q,k)). For all non-negative p, q, k the quantity $A(p,q,k)$ verifies the following properties,

1. $A(p,q,k) = -A(q,p,k)$;
2. $A(p,q,k) = 0$, if $p_i > q_i + 1 + k_i$ or $q_i > p_i + 1 + k_i$ for some $i = 1, 2$;
3.

$$\begin{aligned}
A(p,q,k)^2 &\lesssim \frac{p!q!k!}{(p+q-1-k)!^2}\left[\frac{(p_1-q_1+1+k_1)(q_2-p_2+1+k_2)-(p_2-q_2+1+k_2)(q_1-p_1+1+k_1)}{(p-q+1+k)!(q-p+1+k)!}\right]^2\\
&= \frac{p!q!k!}{(p+q-1-k)!^2}\left[\frac{1}{(p_2-q_2+1+k_2)(q_1-p_1+1+k_1)}-\frac{1}{(p_1-q_1+1+k_1)(q_2-p_2+1+k_2)}\right]^2\\
&\quad\times\frac{1}{(q-p+k)!^2(p-q+k)!^2}\\
&\lesssim \frac{p!q!k!}{(p+q-1-k)!^2(q-p+k)!^2(p-q+k)!^2}\lesssim\frac{p!q!}{(p+q-1-k)!^2(q-p-1+k)!^2k!}.
\end{aligned}$$

We can permute the series in the indices k and p that appear in the expression of B^c; moreover from property 2 of $A(p,q,k)$, we deduce that the vorticity equation for (4.3) reads as

$$\frac{d}{dt}\varphi_k(t) = B_k^c(\varphi), \qquad \forall k>0, \tag{4.24}$$

where B_k^c denotes the k-th component in the Hermite basis of B^c. Namely

$$B^c(\varphi) = \sum_{k>0} B_k^c(\varphi)H_k^c(x), \tag{4.25}$$

where

$$B_k^c(\varphi) = -\frac{\sqrt{c}}{|k|}\sum_{p\geq|k|/2}\sum_{\substack{q\geq 0\\|q|<|p|}}(|p|-|q|)A(p,q,k)\varphi_p\varphi_q. \tag{4.26}$$

4.5.1 Regularity

In this section we study the L^r-regularity of B^c and its derivatives with respect to the measures $\mu_{\sigma^c,\gamma}$. These technical results are necessary to prove Theorem 4.6.1 below.

Proposition 4.5.1. *Let $\beta\in\mathbb{R}$ and $\varepsilon>0$; then $B^c\in L^r_{\mu_{\sigma^c,\gamma}}(H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2);H^{\beta}_{\sigma^c}(\mathbb{R}^2))$ for all $r\geq 1$.*

Proof. It is enough to prove $\mathbb{E}_{\mu_{\sigma^c,\gamma}} \|B^c(\varphi)\|^{2r}_{\beta,\sigma^c} < +\infty$ for all odd $r \geq 1$. We have

$$\begin{aligned}
\mathbb{E}_{\mu_{\sigma^c,\gamma}} \|B^c(\varphi)\|^{2r}_{\beta,\sigma^c} &= \mathbb{E}_{\mu_{\sigma^c,\gamma}} \left(\sum_{k>0} (1+c|k|)^\beta |B^c_k(\varphi)|^2 \right)^r \\
&= \mathbb{E}_{\mu_{\sigma^c,\gamma}} \sum_{k_1,\dots,k_r} \prod_{i=1,\dots,r} (1+c|k_i|)^\beta |B^c_{k_i}(\varphi)|^2 \\
&\leq \sum_{k_1,\dots,k_r} \prod_{i=1,\dots,r} (1+c|k_i|)^\beta \left(\mathbb{E}_{\mu_{\sigma^c,\gamma}} |B^c_{k_i}(\varphi)|^{2r} \right)^{1/r} \\
&= \left[\sum_{k>0} (1+c|k|)^\beta \left(\mathbb{E}_{\mu_{\sigma^c,\gamma}} |B^c_k(\varphi)|^{2r} \right)^{1/r} \right]^r .
\end{aligned}$$

From (4.26) we get

$$\begin{aligned}
\mathbb{E}_{\mu_{\sigma^c,\gamma}} |B^c_k(\varphi)|^{2r} &\leq \left[\sum_{p,p'} \sum_{q,q'} \frac{c}{|k|^2} (|p|-|q|)(|p'|-|q'|) A(p,q,k) A(p',q',k) \left(\mathbb{E}_{\mu_{\sigma^c,\gamma}} (\varphi_p \varphi_q \bar{\varphi}_{p'} \bar{\varphi}_{q'})^r \right)^{1/r} \right]^r \\
&= \left[2 \sum_{p,q} \frac{c}{|k|^2} (|p|-|q|)^2 A(p,q,k)^2 \left(\mathbb{E}_{\mu_{\sigma^c,\gamma}} |\varphi_p|^{2r} \right)^{1/r} \left(\mathbb{E}_{\mu_{\sigma^c,\gamma}} |\varphi_q|^{2r} \right)^{1/r} \right]^r \\
&\leq C(r,\gamma,c) \left[\sum_{p,q} \frac{1}{|k|^2} \frac{(|p|-|q|)^2}{(1+c|p|)^2 (1+c|q|)^2} A(p,q,k)^2 \right]^r ,
\end{aligned}$$

where $C(r,\gamma,c) = \left(\frac{4c}{\gamma^2} \right)^r r!^2$. By property 3 of $A(p,q,k)$, we obtain

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}} |B^c_k(\varphi)|^{2r} \leq C(r,\gamma,c) \left[\frac{1}{|k|^2 k!} \sum_{p,q} \frac{p!q!}{(p+q-1-k)!^2 (q-p-1+k)!^2} \right]^r < +\infty, \qquad \forall\ k > 0.$$

We conclude that for all β, $\mathbb{E}_{\mu_{\sigma^c,\gamma}} \|B^c(\varphi)\|^{2r}_{\beta,\sigma^c}$ is bounded by

$$C(r,\gamma,c) \left[\sum_{k>0} \sum_{|p|\geq|k|/2} \sum_{\substack{q\geq 0 \\ |q|<|p|}} \frac{p!q!}{(p+q-1-k)!^2 (q-p-1+k)!^2 k! |k|^2 (1+c|k|)^{-\beta}} \right]^r < \infty.$$

□

In particular the field B^c takes values in the Cameron-Martin space $H^2_{\sigma^c}$.

In Malliavin calculus (c.f.[53]), for a functional F defined on an abstract Wiener space (X, P, H), where P and H denote the corresponding Wiener measure and Cameron-Martin space, one defines derivatives along directions $h \in H$ as follows:

$$D_h F(\omega) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(\omega + h) - F(\omega)],$$

the limit being taken almost everywhere with respect to P. Then these derivatives determine a gradient operator which is a linear operator on H and we can use the identification $\nabla F \in H$ by Riesz theorem. If ∇F is Hilbert-Schmidt we can iterate the procedure and define the second gradient (etc).

Proposition 4.5.2. *Let $\beta \in \mathbb{R}$ and $\varepsilon > 0$; then $\nabla B^c \in L^r_{\mu_{\sigma^c,\gamma}}(H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2); H.S.(H^2_{\sigma^c}(\mathbb{R}^2); H^{\beta}_{\sigma^c}(\mathbb{R}^2)))$ and*
$\nabla^2 B^c \in L^r_{\mu_{\sigma^c,\gamma}}(H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2); H.S.(H^2_{\sigma^c}(\mathbb{R}^2) \otimes H^2_{\sigma^c}(\mathbb{R}^2); H^{\beta}_{\sigma^c}(\mathbb{R}^2)))$ for all $r \geq 1$.

Proof. First we compute the Malliavin derivative of $B^c(\varphi)$ with respect to the j-th order Hermite polynomial, $H^c_j \in H^2_{\sigma^c}$; we have

$$D_{H^c_j} B^c(\varphi) = \sum_{k>0} D_{H^c_j} B^c_k(\varphi) H^c_k(x),$$

where by the definition above

$$D_{H^c_j} B^c_k(\varphi) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[B^c_k(\varphi + \varepsilon H^c_j) - B^c_k(\varphi)]$$

and the above limit is taken almost everywhere with respect to $\mu_{\sigma^c,\gamma}$. Therefore

$$\begin{aligned} D_{H^c_j} B^c_k(\varphi) &= -\frac{\sqrt{c}}{|k|} \left(\sum_{\substack{q \geq 0 \\ |q|<|j|}} (|j| - |q|) A(j,q,k) \varphi_q + \sum_{|p|>|j|} (|p| - |j|) A(p,j,k) \varphi_p \right) \\ &= -\frac{\sqrt{c}}{|k|} \sum_{q \geq 0} (|j| - |q|) A(j,q,k) \varphi_q, \end{aligned}$$

in the last equality we relabelled the series in p and used property 1 of $A(p,q,k)$ (c.f. Remark 4.5.1). Also we have

$$D_{H^c_i} D_{H^c_j} B^c_k(\varphi) = -\frac{\sqrt{c}}{|k|} (|j| - |i|) A(j,i,k).$$

We denote by $\{\hat{H}^c_k(x)\}_k$ the orthonormal basis of $H^2_{\sigma^c}$, that is $\hat{H}^c_k(x) = \frac{H^c_k(x)}{1+c|k|}$ for all $k \geq 0$, and we have

$$\|\nabla B^c(\varphi)\|^{2r}_{H.S.(H^2_{\sigma^c}; H^{\beta}_{\sigma^c})} = \left(\sum_{j \geq 0} \|D_{\hat{H}^c_j} B^c(\varphi)\|^2_{\beta,\sigma^c} \right)^r = \left(\sum_{j,k} \frac{|D_{H^c_j} B^c_k(\varphi)|^2}{(1 + c|k|)^{-\beta}(1 + c|j|)^2} \right)^r$$

and

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|\nabla B^c(\varphi)\|^{2r}_{H.S.(H^2_{\sigma^c};H^\beta_{\sigma^c})} = \mathbb{E}_{\mu_{\sigma^c,\gamma}} \sum_{\substack{j_1,\dots,j_r \\ k_1,\dots,k_r}} \prod_{i=0}^{r} \frac{|D_{H_{j_i}} B^c_{k_i}(\varphi)|^2}{(1+c|k_i|)^{-\beta}(1+c|j_i|)^2}$$
$$\leq \left[\sum_{j,k} \frac{\left(\mathbb{E}_{\mu_{\sigma^c,\gamma}}|D_{H^c_j}B^c(\varphi)|^{2r}\right)^{1/r}}{(1+c|k|)^{-\beta}(1+c|j|)^2}\right]^r$$

where

$$\left(\mathbb{E}_{\mu_{\sigma^c,\gamma}}|D_{H^c_j}B^c(\varphi)|^{2r}\right)^{1/r} \leq \sum_{q\geq 0} \frac{c(|j|-|q|)^2}{|k|^2} A(j,q,k)^2 \left(\mathbb{E}_{\mu_{\sigma^c,\gamma}}|\varphi_q|^{2r}\right)^{1/r}$$
$$= \frac{2r!^{1/r}}{\gamma} \sum_{q\geq 0} \frac{c(|j|-|q|)^2}{|k|^2} A(j,q,k)^2 \frac{1}{(1+c|q|)^2}.$$

By property 3 of $A(p,q,k)$ and for every $\beta \in \mathbb{R}$

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|\nabla B^c(\varphi)\|^{2r}_{H.S.(H^2_{\sigma^c};H^\beta_{\sigma^c})} \leq \left(\frac{2c}{\gamma}\right)^r r! \left[\sum_{j,k,q} \frac{(|j|-|q|)^2}{(1+c|k|)^{-\beta}(1+c|j|)^2|k|^2(1+c|q|)^2} A(j,q,k)^2\right]^r$$
$$< \infty.$$

In particular we have

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|\nabla B^c(\varphi)\|^{r}_{H.S.(H^2_{\sigma^c};H^\beta_{\sigma^c})} \leq \tilde{C}(\gamma,c)^r r!^{1/2} \tag{4.27}$$

Similarly, for the second order derivative we have

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|\nabla^2 B^c(\varphi)\|^{2r}_{H.S.(H^2_{\sigma^c}\otimes H^2_{\sigma^c};H^\beta_{\sigma^c})} \leq \left[\sum_{i,j,k} \frac{\left(\mathbb{E}_{\mu_{\sigma^c,\gamma}}|D_{H^c_i}D_{H^c_j}B^c(\varphi)|^{2r}\right)^{1/r}}{(1+c|k|)^{-\beta}(1+c|j|)^2(1+c|i|)^2}\right]^r$$
$$= \left[\sum_{i,j,k} \frac{c(|j|-|i|)^2 A(j,i,k)^2}{|k|^2(1+c|k|)^{-\beta}(1+c|j|)^2(1+c|i|)^2}\right]^r < +\infty.$$

□

4.5.2 The divergence operator

The divergence on an abstract Wiener space (X,P,H) is the dual of the gradient operator on this space. Namely, for $Z : X \to H$ the divergence of Z, denoted by $\mathrm{div}_P Z$, is such that

$$E_P(F\mathrm{div}_P Z) = -E_P < Z, \nabla F >$$

for every functional F in $L^2_P(X)$ with $\nabla F \in L^2_P(X;H)$.

For all n we denote by $B^{n,c}$ a Galerkin approximation of B^c, that is the projection of B^c on the subspace of $L^2_{\sigma^c}$ generated by $\{H_{\alpha_1}, \dots, H_{\alpha_{d(n)}}\}$ where $\alpha_1, \dots, \alpha_{d(n)}$ denote non-negative pairs of $\mathbb{Z}^2$. We have

$$B^{n,c}(\varphi) = \sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} B^{n,c}_k(\varphi) H^c_k(x). \tag{4.28}$$

We denote by $\mu^n_{\sigma^c,\gamma}$ the probability measure given by

$$d\mu^n_{\sigma^c,\gamma}(\varphi) = \prod_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \frac{1}{Z^n_{\gamma,k}} e^{-\frac{\gamma}{2}(1+c|k|)^2|\varphi_k|^2} d\varphi_k,$$

and by η^n_γ the Radon-Nikodym density of $d\mu^n_{\sigma^c,\gamma}$ with respect to the Lebesgue measure, $d\lambda^n$, that is $\eta^n_\gamma = d\mu^n_{\sigma^c,\gamma}/d\lambda^n = \frac{1}{Z^n_\gamma} e^{-\frac{\gamma}{2}\sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}}(1+c|k|)^2|\varphi_k|^2}$, where $\frac{1}{Z^n_\gamma} = \prod_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \frac{\gamma(1+c|k|)^2}{2\pi}$.

The divergence of $B^{n,c}$ with respect to the measure $\mu^n_{\sigma^c,\gamma}$ is given by

$$\mathrm{div}_{\mu^n_{\sigma^c,\gamma}} B^{n,c}(\varphi) = \mathrm{div} B^{n,c}(\varphi) + < B^{n,c}_k(\varphi), \frac{\nabla \eta^n_\gamma}{\eta^n_\gamma} >_{\mathbb{C}^{d(n)}} .$$

On one hand

$$\begin{aligned}\mathrm{div} B^{n,c}(\varphi) &= \sum_k D_{H^c_k} B^{n,c}_k(\varphi) = \sum_k \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[B^{n,c}_k(\varphi+\varepsilon H^c_k) - B^{n,c}_k(\varphi)\right] \\ &= -\sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \frac{\sqrt{c}}{|k|}\sum_{p\geq 0}(|p|-|k|)A(p,k,k)\varphi_p;\end{aligned}$$

on the other

$$\begin{aligned}< B^{n,c}_k(\varphi), \frac{\nabla\eta^n_\gamma}{\eta^n_\gamma} >_{\mathbb{C}^{d(n)}} &= -\gamma \sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} (1+c|k|)^2 B^{n,c}_k(\varphi)\bar{\varphi}_k \\ &= \gamma \sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \frac{\sqrt{c}(1+c|k|)^2}{|k|} \sum_{|p|\geq|k|/2} \sum_{\substack{q\geq 0\\ |q|<|p|}} (|p|-|q|)A(p,q,k)\varphi_p\varphi_q\bar{\varphi}_k.\end{aligned}$$

Therefore,

$$\operatorname{div}_{\mu^n_{\gamma,\sigma^c}} B^{n,c}(\varphi) = -\sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \frac{\sqrt{c}}{|k|}\sum_{p\geq 0}(|p|-|k|)A(p,k,k)\varphi_p \tag{4.29}$$

$$+\gamma \sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \frac{\sqrt{c}(1+c|k|)^2}{|k|} \sum_{|p|\geq |k|/2} \sum_{\substack{q\geq 0\\ |q|<|p|}} (|p|-|q|)A(p,q,k)\varphi_p\varphi_q\bar{\varphi}_k. \tag{4.30}$$

We study the L^r-regularity of $\operatorname{div}_{\mu_{\sigma^c,\gamma}} B^c$ with respect to the measures $\mu_{\sigma^c,\gamma}$, also this result is necessary to prove Theorem 4.6.1 below.

Proposition 4.5.3. *Let $\beta\in\mathbb{R}$ and $\varepsilon>0$, then for all $r\geq 1$ we have $\operatorname{div}_{\mu_{\sigma^c,\gamma}} B^c \in L^r_{\mu_{\sigma^c,\gamma}}(H^{-\varepsilon}_{\sigma^c};\mathbb{R})$.*

Proof. We show that

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}}|\operatorname{div}_{\mu_{\sigma^c,\gamma}} B^c|^{2r} < +\infty$$

for all odd $r\geq 1$. We have

$$\begin{aligned}
\Big[\mathbb{E}_{\mu_{\sigma^c,\gamma}}|\operatorname{div}_{\mu_{\sigma^c,\gamma}} B^c|^{2r}\Big]^{1/2r} &\leq \left[\mathbb{E}_{\mu_{\sigma^c,\gamma}}\Big|\sum_k \frac{\sqrt{c}}{|k|}\sum_{p\geq 0}(|p|-|k|)A(p,k,k)\varphi_p\Big|^{2r}\right]^{1/2r}\\
&+\left[\mathbb{E}_{\mu_{\sigma^c,\gamma}}\Big|\sum_k \frac{\sqrt{c}(1+c|k|)^2}{|k|}\sum_{|p|\geq|k|/2}\sum_{\substack{q\geq 0\\ |q|<|p|}}(|p|-|q|)A(p,q,k)\varphi_p\varphi_q\bar{\varphi}_k\Big|^{2r}\right]^{1/2r}\\
&\leq \left[\sum_{k,p}\frac{c}{|k|^2}(|p|-|k|)^2A(p,k,k)^2(\mathbb{E}_{\mu_{\sigma^c,\gamma}}|\varphi_p|^{2r})^{1/r}\right]^{1/2}\\
&+\left[\sum_{k,p,q}\frac{c(1+c|k|)^4}{|k|^2}(|p|-|q|)^2A(p,q,k)^2(\mathbb{E}_{\mu_{\sigma^c,\gamma}}(\varphi_p\varphi_q\bar{\varphi}_k)^{2r})^{1/r}\right]^{1/2}\\
&\leq \frac{2}{\gamma}r!^{\frac{1}{2r}}\left[\sum_{k,p}\frac{c}{|k|^2}\frac{(|p|-|k|)^2}{(1+c|p|)^2}A(p,k,k)^2\right]^{1/2}\\
&+\frac{2}{\gamma}^{3/2}(3r)!^{\frac{1}{2r}}\sqrt{c}\left[\sum_{k,p,q}\frac{(1+c|k|)^4}{|k|^2}\frac{(|p|-|q|)^2}{(1+c|p|)^2(1+c|q|)^2(1+c|k|)^2}A(p,q,k)^2\right]^{1/2}\\
&<\infty.
\end{aligned}$$

In particular we get

$$\left[\mathbb{E}_{\mu_{\sigma^c,\gamma}}|\mathrm{div}_{\mu_{\sigma^c,\gamma}}B^c|^r\right]^{1/r} \leq C(\gamma,c)r!^{\frac{1}{2r}}. \tag{4.31}$$

□

4.6 Existence and quasi-invariance

In this section, we prove that there exists a flow for the vector field B^c defined almost everywhere with respect to each probability measure $\mu_{\sigma^c,\gamma}$. Moreover, we show that the probability measures $\mu_{\sigma^c,\gamma}$ are quasi-invariant with respect to these flows.

The proof of these facts will follow from a result by A. S. Ustunel, Theorem 5.3.1 of [70]. This theorem gives some exponential integrability conditions on the vector field that ensure existence and quasi-invariance, generalizing the previous result in [30]. Both results hold for vector fields on Wiener spaces taking values in the Cameron-Martin spaces, thus below we fix $\beta = 2$.

Recall that, if ν is a measure defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $T : \mathrm{supp}_\nu \to \mathrm{supp}_\nu$, we say that ν is quasi-invariant under T if $T * \nu << \nu$.

Theorem 4.6.1. *Let $\beta = 2$ and $\varepsilon > 0$, then $B^c : H^{-\varepsilon}_{\sigma^c} \to H^2_{\sigma^c}$ is such that there exists an almost surely unique flow for B^c defined by*

$$U^c_t(\varphi) = \varphi + \int_0^t B^c(U^c_s(\varphi))ds \qquad \mu_{\sigma^c,\gamma} - a.e.\ \varphi \in H^{-\varepsilon}_{\sigma^c} \cap L^p_{loc}, \quad \forall t \in \mathbb{R}. \tag{4.32}$$

Moreover, the measure $\mu_{\sigma^c,\gamma}$ is quasi-invariant under U^B_t and

$$k_t(\varphi) = \exp\left(\int_0^t \mathrm{div}_{\mu_{\sigma^c,\gamma}}B^c(U^c_{-s}(\varphi))ds\right) \tag{4.33}$$

is the corresponding Radon-Nikodym density, defined by $k_t := \frac{dU^c_t * \mu_{\sigma^c,\gamma}}{d\mu_{\sigma^c,\gamma}}$. We have $k_t \in L^r_{\mu_{\sigma^c,\gamma}}$, for all $r \geq 1$.

Proof. We know from Propositions 4.5.1 to 4.5.3 that $B^c \in L^r_{\mu_{\sigma^c,\gamma}}(H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2); H^\beta_{\sigma^c}(\mathbb{R}^2))$; $\nabla B^c \in L^r_{\mu_{\sigma^c,\gamma}}(H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2); H.S.(H^2_{\sigma^c}(\mathbb{R}^2); H^\beta_{\sigma^c}(\mathbb{R}^2)))$; and that $\mathrm{div}_{\mu_{\sigma^c,\gamma}}B^c \in L^r_{\mu_{\sigma^c,\gamma}}(H^{-\varepsilon}_{\sigma^c}; \mathbb{R})$, for all $r \geq 1$. In order to apply Ustunel's result we only have to prove that for any given $t \in \mathbb{R}$, there exists a positive λ such that

$$\int_0^t \mathbb{E}_{\mu_{\sigma^c,\gamma}}\left[\exp(\lambda|\mathrm{div}_{\mu_{\sigma^c,\gamma}}B^c(\varphi)|) + \exp(\lambda\|\nabla B^c(\varphi)\|)\right] < +\infty, \tag{4.34}$$

where $\|\nabla B^c(\varphi)\|$ is the operator norm given by $\sup_{\substack{h\in H^2_{\sigma^c}\\ |h|\leq 1}} \|D_h B^c(\varphi)\|_{2,\sigma^c}$.

We use estimatives (4.31) and (4.27) to get respectively

$$\begin{aligned}\int_0^t \mathbb{E}_{\mu_{\sigma^c,\gamma}}\left[\exp(\lambda|\mathrm{div}_{\mu_{\sigma^c,\gamma}} B^c(\varphi)|)\right] &= \int_0^t \sum_{j\geq 0}\frac{\lambda^j}{j!}\mathbb{E}_{\mu_{\sigma^c,\gamma}}|\mathrm{div}_{\mu_{\sigma^c,\gamma}} B^c(\varphi)|^j \\ &\leq |t|\sum_{j\geq 0}\frac{(\lambda C(\gamma,c))^j}{\sqrt{j!}} < +\infty, \quad \forall\lambda>0,\end{aligned}$$

and

$$\begin{aligned}\int_0^t \mathbb{E}_{\mu_{\sigma^c,\gamma}}\left[\exp(\lambda\|\nabla B^c(\varphi)\|)\right] &\leq \int_0^t \mathbb{E}_{\mu_{\sigma^c,\gamma}}\left[\exp(\lambda\|\nabla B^c(\varphi)\|_{H.S.})\right] \\ &= \int_0^t \sum_{j\geq 0}\frac{\lambda^j}{j!}\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|\nabla B^c(\varphi)\|^j_{H.S.} \\ &\leq |t|\sum_{j\geq 0}\frac{(\lambda \tilde{C}(\gamma,c))^j}{\sqrt{j!}} < +\infty, \quad \forall\lambda>0.\end{aligned}$$

We conclude the proof since all the hypothesis of Theorem 5.3.1 of [70] are satisfied. In the work [30], under these assumptions, it is proved that $k_t \in L^r_{\mu_{\sigma^c,\gamma}}$ for all $r\geq 1$.

$\square$

Continuity. The flow is continuous from $H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2)$ to $H^2_{\sigma^c}(\mathbb{R}^2)$ on the support of $\mu_{\sigma^c,\gamma}$ for all times. We write

$$\begin{aligned}\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|U^c_t(\varphi_1)-U^c_t(\varphi_2)\|_{2,\sigma^c} &\leq \mathbb{E}_{\mu_{\sigma^c,\gamma}}\|U^c_t(\varphi_1)-U^{c,n}_t(\varphi_1)\|_{2,\sigma^c} \\ &+ \mathbb{E}_{\mu_{\sigma^c,\gamma}}\|U^{c,n}_t(\varphi_1)-U^{c,n}_t(\varphi_2)\|_{2,\sigma^c} \\ &+ \mathbb{E}_{\mu_{\sigma^c,\gamma}}\|U^{c,n}_t(\varphi_2)-U^c_t(\varphi_2)\|_{2,\sigma^c}.\end{aligned}$$

where $U^{c,n}_t$ denotes a finite dimensional approximation of $U^{c,n}_t$. On one hand there exists $n\geq\max\{n_1,n_2\}$ for $n_1,n_2\in\mathbb{N}$ such that

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|U^c_t(\varphi_1)-U^{c,n}_t(\varphi_1)\|_{2,\sigma^c} + \mathbb{E}_{\mu_{\sigma^c,\gamma}}\|U^{c,n}_t(\varphi_2)-U^c_t(\varphi_2)\|_{2,\sigma^c} \leq \frac{2\varepsilon}{3}.$$

On the other hand, for a fixed $n\geq\max\{n_1,n_2\}$, the flow $U^{c,n}_t$ is continuous for being the flow associated to the quadratic vector field $B^{c,n}$. We conclude that there exists $\delta>0$, such that for all $\varphi_1,\varphi_2\in H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2)$ satisfying $\|\varphi_1-\varphi_2\|_{-\varepsilon,\sigma^c}\leq\delta$, we have

$$\mathbb{E}_{\mu_{\sigma^c,\gamma}}\|U^c_t(\varphi_1)-U^c_t(\varphi_2)\|_{2,\sigma^c}\leq\varepsilon.$$

Finally, we recover the velocity $\tilde{u}$ and the pressure p. On one hand

$$\tilde{u}=\sigma^c\nabla^\perp U^c_t(\varphi), \qquad \mu_{\sigma^c,\gamma}-a.e.\ \varphi\in H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2)\cap L^p_{loc}(\mathbb{R}^2), \tag{4.35}$$

on the other, by taking the divergence of equation (4.1), we obtain

$$(L^c - 2cI)p = -\mathrm{div}\,\mathrm{tr}\,|\nabla\tilde{u}|^2.$$

The operator $(L^c - 2cI)$ is invertible since the value $2c$ doesn't belong to the spectrum of L^c. Moreover, we computed in Subsection 4.3 the integral kernel of the inverse of L^c, see equations (4.16)-(4.17), from this and by a perturbative argument it is possible to get the integral kernel of $(L^c - 2cI)^{-1}$. Hence we have

$$p = -(L^c - 2cI)^{-1}\mathrm{div}\,\mathrm{tr}\,|\nabla\nabla^{\perp}U_t^c(\varphi)|^2, \qquad \mu_{\sigma^c,\gamma} - a.e.\ \varphi \in H^{-\varepsilon}_{\sigma^c}(\mathbb{R}^2) \cap L^p_{loc}(\mathbb{R}^2). \tag{4.36}$$

Remark 4.6.1. Last we observe that in the limit when the parameter c tends to zero, equations (4.1) converge to the "standard" incompressible Euler equations, thus we can formally look at (4.35) and (4.36) as approximations of the solutions for these equations. However, we cannot rigorously consider such limit since the measures $\mu_{\sigma^c,\gamma}$ for each c are mutually singular.

Chapter 5

Asymptotics for the stochastic Navier-Stokes equations

5.1 Introduction

The Navier-Stokes equations describe the motion of an incompressible viscous flow

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + \varepsilon \Delta u - \nabla p, \qquad \nabla \cdot u = 0,$$

where $\varepsilon > 0$ denotes the viscosity coefficient. We consider these equations on the two-dimensional torus $\mathbb{T}^2 \simeq [0, 2\pi] \times [0, 2\pi]$, that is with prescribed periodic boundary conditions, then the mean velocity of the flow is denoted by $u : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}^2$ and the pressure by $p : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$. The bibliography about these equations is very vast, we refer to the works of R. Temam [67] and to ones of A. Majda and A. Bertozzi [51].

Let $\{e_k\}_{k\in\mathbb{Z}^2}$ be the orthonormal basis of $L^2(\mathbb{T}^2)$ given by $e_k = e^{ik\cdot x}$ for all positive $k \in \mathbb{Z}^2$ (we say that $k \in \mathbb{Z}^2$ is positive if $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$). Then we can write the equations in Fourier modes,

$$\frac{d}{dt} u_k = B_k(u) - \varepsilon |k|^2 u_k, \qquad \forall\ k \in \mathbb{Z}^2$$

where B and B_k denote the Euler vector field and its k-th component in the given basis. That is $B(u) = \sum_{k>0} B_k(u) e_k(x)$ where

$$B_k(u) = \sum_{k_1+k_2=k} u_{k_1}(ik_1) \cdot u_{k_2}. \tag{5.1}$$

We remark that this last expression is equivalent to the one given in [3] (that is (3.5)-(3.6) when the period is 2π).

5.1.1 Stochastic Navier-Stokes equations

Now consider a stochastic perturbation of the Navier-Stokes equations

$$du_k = \sum_{k_1+k_2=k} u_{k_1}(ik_1)\cdot u_{k_2}dt - \varepsilon|k|^2 u_k + d\beta_k(t),$$

where $\{\beta_k\}_{k\in\mathbb{Z}^2}$ is a sequence of i.i.d. complex-valued Brownian motions. Within the firsts to consider a stochastic version of the Navier-Stokes equations we refer to A. Bensoussan and R. Temam in 1973 [16]; M. Capiński and N. Cutland in 1992 [26]; Z. Brzeźniak, M. Capiński and F. Flandoli still in 1992 [20]. For these equations F. Flandoli studied dissipativity and invariant measures [42] and joint with D. Gatarek the existence of martingale and stationary solutions [43]. Ergodicity for the three-dimensional stochastic Navier-Stokes equations was studied by G. Da Prato and A. Debussche [60] cf. also [32]. Existence of global L^2-solutions was proved by R. Mikulevicius and B. L. Rozovskii [56].

Passing to vorticity variables $\omega_k^\varepsilon = ik^\perp\cdot u_k$ such that $u_k = -i\frac{k^\perp}{|k|^2}\omega_k^\varepsilon$, we get the vorticity formulation of the equations,

$$d\omega_k^\varepsilon = \sum_{k_1+k_2=k} \frac{k_1\cdot k_2^\perp}{|k_2|^2}\omega_{k_1}^\varepsilon\omega_{k_2}^\varepsilon dt - |k|^2\omega_k^\varepsilon dt + ik^\perp\cdot d\beta_k.$$

We remark that there exists no invariant measure for the deterministic Navier-Stokes equation, since the system is not conservative.

Now, let L be the infinite-dimensional Ornstein-Uhlenbeck operator defined by

$$Lf = \sum_k \frac{1}{|k|^2}D_{e_k}^2 f - \varepsilon|k|^2 u_k D_{e_k} f \qquad \forall\ f\in\mathcal{D}$$

and μ the Gaussian probability measure given by

$$\mu(d\omega^\varepsilon) = Z^{-1}e^{-\frac{1}{2}\sum_k \omega_k^\varepsilon\omega_{-k}^\varepsilon}\Pi_k d\omega_k^\varepsilon.$$

We remark that the covariance which defines μ is conserved by the Euler drift, indeed

$$\frac{1}{2}\sum_k \omega_k^\varepsilon\omega_{-k}^\varepsilon = \frac{1}{2}\sum_k |\omega_k^\varepsilon|^2 = \frac{1}{2}\int_{\mathbb{T}^2}|\text{curl } u|^2 dx$$

which corresponds to the "enstrophy" $S(u)$ as defined in Subsection 1.2 from [3]. It is also proved there that $\frac{d}{dt}S(u) = 0$. Also we recall that the measure μ is invariant under the Euler velocity flow; its support is the Sobolev space H^β for values of $\beta < 0$ and that the Euler vector field is L^p_μ-integrable for all $p \geq 1$ when $\beta < -1$ (see [3] and [27]).

In the following proposition we show that the Gaussian measure μ is infinitesimally invariant for the Ornstein-Uhlenbeck operator L.

Proposition 5.1.1. *For all $f\in\mathcal{D}$, $\int Lf d\mu = 0$.*

Proof. For all $n \in \mathbb{N}$ and for all test function $f \in \mathcal{D}$ we have

$$\begin{aligned}\int L^n f(u)d\mu(u) &= \sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \int \left(\frac{1}{|k|^2}\mathcal{D}^2_{e_k} f(u) - \varepsilon |k|^2 u_k \mathcal{D}_{e_k} f(u)\right) d\mu(u) \\ &= \sum_{k\in\{\alpha_1,\dots,\alpha_{d(n)}\}} \int \left[-\frac{1}{|k|^2}\left(-\frac{\varepsilon}{2}\mathcal{D}_{e_k}\|u\|_2^2\right)\mathcal{D}_{e_k} f(u) - \varepsilon |k|^2 u_k \mathcal{D}_{e_k} f(u)\right] d\mu(u),\end{aligned}$$

where $\{\alpha_1, \dots, \alpha_{d(n)}\} \in \mathbb{Z}^2$. On the other hand

$$\begin{aligned}\frac{\varepsilon}{2}\mathcal{D}_{e_k}\|u\|_2^2 &= \frac{\varepsilon}{2}\lim_{\eta\downarrow 0}\frac{\|u+\eta e_k\|^2 - \|u\|_2^2}{\eta} \\ &= \frac{\varepsilon}{2}\lim_{\eta\downarrow 0}\frac{\sum_h h^4|(u+\eta e_k)_h|^2 - \sum_h h^4|u_h|^2}{\eta} \\ &= \frac{\varepsilon}{2}\lim_{\eta\downarrow 0}\frac{k^4|u_k+\eta|^2 - k^4|u_k|^2}{\varepsilon} = \varepsilon|k|^4 u_k,\end{aligned}$$

where the above limits are taken μ-a.e. We get the conclusion by replacing $\frac{\varepsilon}{2}\mathcal{D}_{e_k}\|u\|_2^2 = \varepsilon|k|^4 u_k$. □

Also we define the following differential operator

$$Af(u) := Lf(u) + \sum_k B_k(u) D_{e_k} f(u),$$

that can be regarded as the infinitesimal generator of a stochastic Navier-Stokes equation with perturbation given by a normalized cylindric Brownian motion,

$$B_t = \sum_k \frac{\beta_k}{|k|} e_k,$$

where β_k are i.i.d. complex-valued Brownian motions. Because of Proposition 5.1.1 and the fact that $\mathrm{div}_\mu B = 0$ (Lemma 2.1.2 from [3]), we have

Proposition 5.1.2. *For all $f \in \mathcal{D}$, $\int Af d\mu = 0$.*

Proof.

$$\forall f \in \mathcal{D}, \quad \int Af d\mu = \int \left(Lf + \sum_k B_k D_{e_k} f\right) d\mu = \int Lf d\mu - \int \mathrm{div}_\mu B f d\mu = 0.$$

□

The infinitesimal invariance of μ under the evolution prescribed by A is sufficient to prove the existence of a stochastic Navier-Stokes flow with similar techniques to the ones used in the case of Euler, this fact is the content of Theorem 3.2.1 from [3] and we recall its statement below.

Theorem 5.1.1. *There exists a stochastic process $\omega^\varepsilon \in C(\mathbb{R}^+; H^\beta)$, such that, for any initial value $x \in H^\beta$ and writing $\omega^\varepsilon = \sum_k \omega_k^\varepsilon e_k$, we have:*

$$\omega_k^\varepsilon = x + \beta_k - \int_0^t [\varepsilon |k|^2 \omega_k^\varepsilon - B_k(\omega^\varepsilon)] ds, \qquad \mu - a.e. x \in H^\beta, \forall\ t \in \mathbb{R}^+,$$

where $B_t = \sum_k \frac{\beta_k(t)}{|k|} e_k$ is a Brownian motion on H^1. Moreover, μ is invariant for ω^ε in the sense that

$$\int \mathbb{E}_x f(\omega^\varepsilon) d\mu(x) = \int f d\mu, \qquad \forall\ t \in \mathbb{R}^+, \forall\ f \in \mathcal{D}.$$

5.2 Weak convergence

Here we want to understand how the asymptotics of the stochastic Navier-Stokes flow behave for vanishing viscosities. To do this we rescale the vorticity coefficient ε such that the stochastic Navier-Stokes flow is now

$$d\omega_k^\varepsilon = -\varepsilon |k|^2 \omega_k^\varepsilon dt + B_k(\omega^\varepsilon) dt + \sqrt{\varepsilon} i k^\perp \cdot d\beta_k(t)$$

and the Gaussian measure

$$\mu(d\omega^\varepsilon) = Z^{-1} e^{-\frac{1}{2} \sum_k \omega_k^\varepsilon \omega_{-k}^\varepsilon} \Pi_k d\omega_k^\varepsilon$$

remains invariant under the evolution when the equation is started from $x \in H^\beta$ for $\beta < -1$ (Proposition 5.1.2), but is now independent from ε.

Lemma 5.2.1. *For $\beta < -3$, we have $\mathbb{E}_\mu \mathbb{E}_x \sup_{t \in [0,T]} \|\omega^\varepsilon\|_\beta \leq C(T)$ uniformly in ε.*

Proof. We have

$$\mathbb{E}_\mu \mathbb{E}_x \sup_t \|\omega^\varepsilon\|_\beta \leq C(T) \left(\mathbb{E}_\mu \mathbb{E}_x \|\omega^\varepsilon(0)\|_\beta + \int_0^T \mathbb{E}_\mu \mathbb{E}_x \|\varepsilon \sum_k |k|^2 \omega_k^\varepsilon\|_\beta ds + \int_0^T \mathbb{E}_\mu \mathbb{E}_x \|B(\omega^\varepsilon)\|_\beta ds \right.$$
$$\left. + \mathbb{E}_\mu \mathbb{E}_x \sup_t \left\| \int_0^t \sum_k \sqrt{\varepsilon} i k^\perp \cdot d\beta_k(s) \right\|_\beta \right) =: I + II + III + IV.$$

Since $\beta < -3$ we have

$$I = \int_{H^\beta} \|x\|_\beta d\mu < \infty$$

and by the invariance property (statement (ii) of Theorem 3.2.1. from [3])

$$II = \varepsilon \int_0^T \mathbb{E}_\mu \|x\|_{\beta+2} ds \leq \varepsilon T \left(\sum_k |k|^{2\beta+4} \right)^{\frac{1}{2}} < \infty.$$

Also, we know from Theorem 2.1. of [27] that

$$III = \int_0^T \mathbb{E}_\mu \|B(x)\|_\beta < \infty.$$

Last, we have

$$IV \leq \varepsilon T \left(\sum_k |k|^{2\beta+2} \right)^{\frac{1}{2}} < \infty.$$

Since ε is small, we conclude

$$\mathbb{E}_\mu \mathbb{E}_x \sup_t \|\omega^\varepsilon\|_\beta \leq \varepsilon C(T) \leq C(T).$$

□

We denote by ν^ε the law of ω^ε on $C(\mathbb{R}^+; H^\beta)$, that is

$$\nu^\varepsilon(\Gamma) = \mathbb{P} \times \mu \left(\{(x,w) \ : \ \omega^\varepsilon(\cdot, x, w) \in \Gamma\} \right), \qquad \Gamma \in \mathcal{B}(C(\mathbb{R}^+; H^\beta))^1.$$

Finally we prove that we can extract a converging subsequence from $\{\nu^\varepsilon\}_{\varepsilon>0}$.

Proposition 5.2.1. *Let $\beta < -3$. The set $\{\nu^\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(C(\mathbb{R}^+; H^\beta))$*[2] *is precompact.*

Proof. By Prohorov's theorem, we get the conclusion if the following hold:

1. $$\lim_{R\to\infty} \sup_{\varepsilon>0} \nu^\varepsilon(\|y(0)\|_\beta \geq R) = 0;$$

2. $$\lim_{\delta\to 0} \sup_{\varepsilon>0} \nu^\varepsilon \left(\sup_{\substack{|t-t'|\leq\delta \\ 0\leq t\leq t'\leq T}} \|y(t) - y(t')\|_\beta \geq \rho \right) = 0, \qquad \forall\ \rho > 0,\ \forall\ T > 0.$$

On one hand

$$\lim_{R\to\infty} \sup_{\varepsilon>0} \nu^\varepsilon \left(\|y(0)\|_\beta \geq R \right) \leq \lim_{R\to\infty} \frac{\mathbb{E}_\mu \|x\|_\beta}{R} \leq \lim_{R\to\infty} \frac{C}{R} = 0.$$

[1] With $\mathcal{B}(C(\mathbb{R}^+; H^\beta))$ we denote all the Borelian sets of $C(\mathbb{R}^+; H^\beta)$.
[2] With $\mathcal{M}(C(\mathbb{R}^+; H^\beta))$ we denote all the measures over $C(\mathbb{R}^+; H^\beta)$.

On the other hand and by Lemma 5.2.1,

$$\begin{aligned}\sup_{\varepsilon>0}\nu^{\varepsilon}\left(\sup_{\substack{|t-t'|\le\delta\\0\le t\le t'\le T}}\|y(t)-y(t')\|_{\beta}\ge\rho\right)&\le\frac{1}{\rho}\sup_{\varepsilon>0}\mathbb{E}_{\mu}\mathbb{E}_{x}\sup_{\substack{|t-t'|\le\delta\\0\le t\le t'\le T}}\|\omega^{\varepsilon}(t)-\omega^{\varepsilon}(t')\|_{\beta}\\&\le\frac{1}{\rho}\sup_{\varepsilon>0}\mathbb{E}_{\mu}\mathbb{E}_{x}\sup_{\substack{|t-t'|\le\delta\\0\le t\le t'\le T}}\left\|\int_t^{t'}d\omega^{\varepsilon}(s)\right\|_{\beta}\\&\le\frac{\delta^{1/2}C(T)}{\rho}\to 0\text{ when }\delta\to 0,\ \forall\ \rho>0,\ \forall\ T>0.\ \square\end{aligned}$$

Appendix A

Desintegration theorem for infinite-dimensional measures

Following [1], we prove that on an abstract Wiener space (X, H, μ) and for $\Phi : X \to \mathbb{R}^n$ a sufficiently regular functional, the conditional measure $\mu(dx|\Phi(x) = \xi)$ (that is on the level sets of Φ) is a Borel measure for all ξ belonging to the interior of the support of $\Phi * \mu$.

A.1 Preliminaries

Consider the abstract Wiener space (X, H, μ), where X is a separable Banach space; H the Cameron-Martin space, that is a separable Hilbert space with norm $|\cdot|_H = \sqrt{(\cdot,\cdot)_H}$ such that H is compactly embedded in X; and μ is the Wiener measure on X. Recall that $\mu(X) = 1$ and $\mu(H) = 0$, thus, for $\Phi : X \to \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, we have $\mu(x \in X|\Phi(x) = \xi) = 0$. The purpose of this appendix is to define a "new" measure with support on the level set $\{x \in X|\Phi(x) = \xi\}$.

A.1.1 The $W^{r,p}$ spaces

Denote by $L^p_\mu(X;\mathbb{R})$ the space of functions $f : X \to \mathbb{R}$ such that $\|f\|_p^p = \int |f|^p d\mu < +\infty$ and by $L_\infty = \bigcap_{p<\infty} L^p_\mu(X;\mathbb{R})$. The space L_∞ is a Fréchet space with respect to the family of norms $\|f\|_p$.

Definition A.1.1. The space $W^{1,p}$ is the space of the maps $f \in L^p_\mu(X;\mathbb{R})$ such that $\nabla f : X \to H$ satisfy $D_h f(x) = (\nabla f(x), h)_H$ for all $h \in H$ and $\nabla f \in L^p_\mu(X;H)$, where $D_h f(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[f(x + \varepsilon h) - f(x)]$ and this limit is taken μ-a.e. $x \in X$.

If $f \in W^{1,p}$, the application $|\nabla f| : X \to \mathbb{R}$ is defined by $|\nabla f|(x) = |\nabla f|_H$. $W^{1,p}$ is a Fréchet space with respect to the norm $\|f\|_p + \|\nabla f\|_p$.

Set $\nabla^2 f(h_1,h_2) = (\nabla(\nabla f(h_1)),h_2)_H$, $\nabla^3 f(h_1,h_2) = (\nabla(\nabla^2 f(h_1,h_2)),h_3)_H$, etc. in order to define the successive derivatives $\nabla^r f(x) \in H_{\otimes^r}$. On the symmetric tensorial product $H_{\otimes^r} = H \otimes \cdots \otimes H$ (r-times) we consider the Hilbert-Schmidt norm.

Definition A.1.2. For every integer $r > 1$, $W^{r,p}$ is the space of functions $f \in W^{r,p}$ such that $D_h f(x) \in W^{r,p}$ for all $h \in H$.

Remark A.1.1. Equivalently, $W^{r,p}$ is the space of functions $f \in L^p_\mu(X;\mathbb{R})$ such that $\nabla^s f \in L^p_\mu(X; HS(H_{\otimes^s},\mathbb{R}))$ for all $1 \le s \le r$. Indeed, if $f : X \to \mathbb{R}$, then $\nabla^s f(x) : H_{\otimes^s} \to \mathbb{R}$ is a linear functional on the Hilbert space $H_{\otimes^s}$; that is $\nabla^s f(x) \in (H_{\otimes^s})'$ and by Riesz representation theorem $(H_{\otimes^s})' \simeq H_{\otimes^s}$. Let $\{e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_s}\}$ be an orthonormal basis of $H_{\otimes^s}$, where $\{e_{k_i}\}_{k_i\in\mathbb{N}}$ is an orthonormal basis of H for $1 \le i \le s$. Then

$$|\nabla^s f(x)|^2_{HS(H_{\otimes^s},\mathbb{R})} = \sum_{k_1,\ldots,k_s} |D_{e_{k_1}} D_{e_{k_2}} \ldots D_{e_{k_s}} f(x)|^2 = \sum_{k_1,\ldots,k_s} |\left(e_{k_1}, D_{e_{k_2}} \ldots D_{e_{k_s}} f(x)\right)_H|^2$$
$$= \cdots = \sum_{k_1,\ldots,k_s} |\left(\nabla^s f(x), e_{k_1} \otimes \ldots \otimes e_{k_s}\right)_{H_{\otimes^s}}|^2 = |\nabla^s f(x)|^2_{H_{\otimes^s}}.$$

Denoting by $|\nabla^r f|(x) = |\nabla^r f(x)|_{H_{\otimes^r}}$, $W^{r,p}$ is a Fréchet space with respect to the norm $\|f\|_p$, $\|\nabla^i f\|_p$ for $1 \le i \le r$, $p = 1,\ldots,\infty$. Define $W_\infty = \bigcap_{p,r} W^{r,p}$. We say that $f = (f_1,\ldots,f_n) \in W_\infty$, if $f_i \in W_\infty$, $1 \le i \le n$.

A.1.2 Capacities and redefinitions

Definition A.1.3. 1. Let $O \subset X$ be an open set. The capacity of O is defined by

$$c_{p,r}(O) := \inf\{\|u\|_{W2r,p};\ u \ge 0, u(x) \ge 1, \mu - \text{a.e. on } O\},$$

2. for $A \subset X$ arbitrary set, the capacity of A is given by

$$c_{p,r}(A) := \{c_{p,r}(O) : O \text{ open } O \supset A\}.$$

3. A set A is said *slim*, if $c_{p,r}(A) = 0, \forall\ p,r \in \mathbb{N}$.

Definition A.1.4. Given a measurable function Φ, we call a (p,r)-*redefinition of* Φ a function Φ^* such that $\Phi = \Phi^*$ a.s. and Φ^* is (p,r)-continuous (that is, if $\forall\varepsilon > 0$ it is possible to find an open set O_ε such that $c_{p,r}(O_\varepsilon) < \varepsilon$ and the restriction of Φ^* to O^c_ε is continuous).

Proposition A.1.1. *For all $\Phi \in W_\infty$ there exists a redefinition Φ^* and a sequence of open sets $\{O_n\}_{n\in\mathbb{N}}$ associated to this redefinition such that*

1. *$\bigcap_n O_n$ is slim,*
2. *Φ^* is continuous on $(\bigcap_n O_n)^c$,*

3. *Φ^* and $\nabla^r \Phi^*$ are continuous on O_n^c (with respect to the uniform convergence on X) for all $n, r \in \mathbb{N}$.*

In what follows and for every μ-integrable function g, $g\mu$ denotes the measure on X with density g with respect to μ and $\Phi * g\mu / \Phi * \mu$ the Radon-Nikodym density of $\Phi * g\mu$ with respect to $\Phi * \mu$.

Definition A.1.5. We say that $\Phi = (\Phi_1, \dots, \Phi_n) \in W_\infty$ is of *maximal rank* and *non-degenerate* if $1/\det \Phi$ is in W_∞, where $[\det \Phi](x) = (\det(\nabla \Phi_i(x), \nabla \Phi_j(x)))^{1/2}$.

We will use the following

Proposition A.1.2. *Whenever $\Phi \in W_\infty$ is of maximal rank and non-degenerate and $g \in W_\infty$, then the measures $\Phi * g\mu$ and $\Phi * \mu$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover, $k(\xi) = d\Phi * \mu / d\xi$ and $k_g(\xi) = d\Phi * g\mu / d\xi$ are $C^\infty(\mathbb{R}^n; \mathbb{R})$.*

The proofs of Propositions A.1.1-A.1.2 can be found in [53].

A.1.3 Inverse image of distributions

From now on fix $\Phi \in W_\infty$ of maximal rank and non-degenerate. For $g \in W_\infty$ we consider the $C^\infty(O; \mathbb{R})$-map

$$< \delta\Phi, g >: \xi \mapsto < \delta_\xi \Phi, g > := k_g(\xi)/k(\xi),$$

where $O = \{\xi \in \mathrm{supp}(\Phi * \mu) \subset \mathbb{R}^n : k(\xi) > 0\}$. In particular the application

$$g \mapsto < \delta\Phi, g >$$

is a continuous linear functional of W_∞ to the space of functions C^∞ on O. If we denote by $S(O)$ the Schwartz space of O and by W' the dual of W_∞ (W' was accurately defined by Watanabe, see [53]) we can consider the dual map

$$\delta_* \Phi : S(O) \to W'$$

that associates linear functionals on W_∞ to distributions over $\mathbb{R}^n$ and such that

$$<< \delta_* \Phi, v >, g > = < v, < \delta\Phi, g >> \tag{A.1}$$

for every $v \in S(O)$ and $g \in W_\infty$.

A.2 Desintegration theorem

We have the following

Theorem A.2.1. *Let $\xi \in O$. There exists a Borelian probability measure on X denoted $\mu(dx|\Phi(x) = \xi)$ with support on $\{x \in X : \Phi(x) = \xi\}$, such that slim sets have measure zero and*

$$< \delta_\xi \Phi, g >= \int g^*(x)\mu(dx|\Phi(x) = \xi)$$

for all $g \in W_\infty$ and g^* any redefinition of g.

The proof of the theorem is given in four separate lemmas.

Lemma A.2.1. *Let $\xi \in O$. The functional $\delta_\xi \Phi$ defined on W_∞ is the limit in W' of a sequence of Borelian measures ν_n^ξ on X.*

Proof. Consider $u_n^\xi(\eta) = C_n$, if $|\eta - \xi| \leq 1/n$; and $u_n^\xi(\eta) = 0$, if $|\eta - \xi| \geq 1/(n-1)$ such that $\int u_n^\xi(\eta)d\eta = 1$. We have that $u_n^\xi(\eta) \in C^\infty(\mathbb{R}^n; \mathbb{R})$ is positive and $u_n^\xi \to \delta_\xi$.

$$\begin{aligned} << \delta_* \Phi, u_n^\xi(\eta)d\eta >, g > &= \int < \delta_\eta \Phi, g > u_n^\xi(\eta)d\eta = \int \frac{k_g(\eta)}{k(\eta)} u_n^\xi(\eta)d\eta \\ &= \int \frac{u_n^\xi(\Phi(x))}{k(\Phi(x))} g(x)d\mu(x), \end{aligned}$$

where we used equation (A.1) in the first equality. Defining $\nu_n^\xi(dx) = \frac{u_n^\xi(\Phi(x))}{k(\Phi(x))}d\mu(x)$, it follows that ν_n^ξ converges to $\delta_\xi \Phi$ in W'. Indeed, for every $g \in W_\infty$, we have

$$< \delta_\xi \Phi, g >= \frac{k_g(\xi)}{k(\xi)} = \lim_{n\to\infty} \int \frac{k_g(\eta)}{k(\eta)} u_n^\xi(\eta)d\eta.$$

□

Lemma A.2.2. *Define*

$$\varphi(x) = \int_0^1 \int_0^1 \frac{|x(t) - x(t')|^{2p}}{|1 - t'|^{1+2p\gamma}} dtdt'$$

where $p \in \mathbb{Z}$, $\gamma \in \mathbb{R}$ are such that $0 < \gamma < \frac{1}{2}$, $2p\gamma > 1$ and $2p(\frac{1}{2} - \gamma) > 1$. Then $\varphi \in W_\infty$ and for $a > 0$ $A_a := \{x \in X : \varphi(x) \leq a\}$ is a compact set with respect to the uniform convergence topology of X. Moreover $\mu\left(X - \bigcup_{a\in\mathbb{N}} A_a\right) = 0$.

Proof. We refer to [1] for the proof of this technical result. □

From Lemma A.2.2 we know that there exist functions in W_∞ with compact support in X. Now, let ψ be the positive and $C^\infty(\mathbb{R};\mathbb{R})$ function defined by $\psi(t)=0$ for $t>1$ and $\psi(t)=1$ for $t\leq\frac{1}{2}$; and, for $a>0$, let $\varphi_a(x):=\psi(\varphi(x)/a)$. By Lemma A.2.2, φ_a belongs to W_∞ and has compact support on X.

Lemma A.2.3. *Let $a>0$ fixed, then*

1. *$\rho_n^a(dx)=\varphi_a(x)\nu_n^\xi(dx)$ has support on A_a and weakly converges to the measure $\nu^{a,\xi}$ on X.*

2. *$\nu^{a,\xi}$ does not charge slim sets and has support on $\{x\in X:\Phi(x)=\xi\}$.*

3. *$\forall g\in W_\infty$,*
$$\int g^*(x)\nu^{a,\xi}(dx)=<\delta_\xi\Phi,g\varphi_a>,$$
where g^* is any redefinition of g.

Proof. 1. By Lemma A.2.2, if $\varphi(x)>a$, then $\varphi_a(x)=0$ and thus ρ_n^a has support on A_a. Let $g\in W_\infty$,
$$\int g(x)\rho_n^a(dx)=\int g(x)\varphi_a(x)\frac{u_n^\xi(\Phi(x))}{k(\Phi(x))}d\mu(x)=\int\frac{k_{g\varphi_a}(\eta)}{k(\eta)}u_n^\xi(\eta)d\eta,$$
from this and Lemma A.2.1 we get
$$\lim_{n\to\infty}\int g(x)\rho_n^a(dx)=<\delta_\xi\Phi,g\varphi_a>,$$
that is ρ_n^a weakly converges to $\nu^{a,\xi}$ with support in the compact A_a. In fact $W_\infty|_{A_a}=\{f\in W_\infty : f \text{ is continuous on } A_a\}$ is dense in the space of bounded continuous functions on A_a.

2. Let A be a slim set of X, that is $\forall\varepsilon>0$ there exists $u_\varepsilon\in W^{2r,p}$ such that $\|u_\varepsilon\|_{W^{2r,p}}<\varepsilon$, $u_\varepsilon(x)\geq 0$ and $u_\varepsilon(x)=1$ μ-a.e.. The inequalities
$$\Phi*(\mathbb{1}_A\mu)\leq\Phi*(u_\varepsilon\mu)=k_{u_\varepsilon}(\eta)d\eta$$
imply
$$\nu^{a,\xi}(A)\leq\left[\frac{\Phi*(\mathbb{1}_A\mu)}{\Phi*(\mu)}\right](\xi)\leq\frac{k_{u_\varepsilon}(\xi)}{k(\xi)}.$$
Since k_{u_ε} can be majorated by $\|u_\varepsilon\|_{W^{2r,p}}$ (see [1] and reference therein), then $k_{u_\varepsilon}\to 0$ for $\varepsilon\to 0$ and $\nu^{a,\xi}$ does not charge slim sets.

3. Consider g^* a redefinition of $g \in W_\infty$ and the associated sequence of open sets $\{O_p\}_{p\in\mathbb{N}}$ such that $\bigcap_p O_p$ is slim. There exists an increasing sequence of functions $\{h_p\}$ continuous on X with support on O_p^c such that $\lim_p h_p = 1$ $\nu^{a,\xi}$-a.e.. We have that g^* is continuous on O_p^c with respect to the norm of the uniform convergence, this fact implies that each function $g^* h_p$ is integrable with respect to the measure $\nu^{a,\xi}$ and
$$\int g^* h_p \nu^{a,\xi}(d\xi) =< \delta_\xi \Phi, g h_p \varphi_a > \leq < \delta_\xi \Phi, g\varphi_a > .$$
Since $g^* = \lim_p g^* h_p$ $\nu^{a,\xi}$-a.e., it follows that g^* is integrable and the Lemma is proved.

∎

Lemma A.2.4. *1. The sequence $\{\nu^{a,\xi}\}_{a\in\mathbb{N}}$ converges weakly through ν^ξ of mass 1.*

2. The measure ν^ξ does not charge slim sets and has support on $\{x \in X : \Phi(x) = \xi\}$.

3. $\forall g \in W_\infty$,
$$\int g^*(x)\nu^\xi(dx) =< \delta_\xi \Phi, g >,$$
where g^* is any redefinition of g.

Proof. 1. From $\mu\left(X - \bigcup_{a\in\mathbb{N}} A_a\right) = 0$ (Lemma A.2.2), we get $\lim_a \varphi_a(x) = 1$ μ-a.e.. Let $v : \mathbb{R}^n \to \mathbb{R}$ be continuous with compact support and $g \in W_\infty$, then we have
$$\begin{aligned}\int v(\xi) < \delta_\xi \Phi, g > d\xi &= \int v(\xi)\left[\frac{k_g(\xi)}{k(\xi)}\right] d\xi = \int \left[\frac{v(\Phi(x))}{k(\Phi(x))}\right] g(x) d\mu(x)\\ &= \lim_a \int \left[\frac{v(\Phi(x))}{k(\Phi(x))}\right] g(x)\varphi_a(x) d\mu(x)\\ &= \lim_a \int v(\xi) < \delta_\xi \Phi, g\varphi_a > d\xi,\end{aligned}$$
that is $< \delta_\xi \Phi, g\varphi_a >$ weakly converges to $< \delta_\xi \Phi, g >$.
However, since $\xi \mapsto < \delta_\xi \Phi, g\varphi_a >$ is continuous and its derivatives are continuous such that the sequence of derivatives are also convergent in L^1_μ, we have that in fact $\lim_a < \delta_\xi \Phi, g\varphi_a >=< \delta_\xi \Phi, g >$ and we deduce
$$\lim_a \int g^*(x)\nu^{a,\xi}(dx) =< \delta_\xi \Phi, g >,$$
that is $\nu^{a,\xi}$ converges weakly to ν^ξ. From $< \delta_\xi \Phi, 1 >= 1$ we deduce that ν^ξ is a probability measure.

2. Similar to the proof of 2 in Lemma A.2.3.

3. Similar to the proof of 3 in Lemma A.2.3.

□

Once we set $\mu(dx|\Phi(x) = \xi) = \nu^\xi$, Theorem A.2.1 is proved.

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